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In any Δ ABC, the following relationship holds :

$$\frac{6ar_a^2}{6+\sqrt{3}a(a+b+c)}+\frac{6br_b^2}{6+\sqrt{3}b(a+b+c)}+\frac{6cr_c^2}{6+\sqrt{3}c(a+b+c)}\ge\frac{324Sr^2}{R(9+20S)-4Sr}$$

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Solution by Soumava Chakraborty-Kolkata-India

WLOG we may assume $a\ge b\ge c$ and then : $r_a^2\ge r_b^2\ge r_c^2$ and

$$\frac{6a}{6+\sqrt{3}a(a+b+c)}\ge\frac{6b}{6+\sqrt{3}b(a+b+c)}\ge\frac{6c}{6+\sqrt{3}c(a+b+c)}$$

$$\therefore\frac{6ar_a^2}{6+\sqrt{3}a(a+b+c)}+\frac{6br_b^2}{6+\sqrt{3}b(a+b+c)}+\frac{6cr_c^2}{6+\sqrt{3}c(a+b+c)}$$

$$\overset{\text{Chebyshev}}{\ge}\frac{1}{3}\left(\sum_{cyc}r_a^2\right)\left(\sum_{cyc}\frac{6a}{6+\sqrt{3}a(a+b+c)}\right)\ge2\left(\sum_{cyc}r_ar_b\right)\left(\sum_{cyc}\frac{1}{\frac{6}{a}+\sqrt{3}(2s)}\right)$$

$$\overset{\text{Bergstrom}}{\ge}2s^2.\frac{9}{6\sum_{cyc}\frac{1}{a}+6\sqrt{3}s}=\frac{18s^2.4S}{\frac{6\sum_{cyc}ab}{R}+24\sqrt{3}sS}\overset{\text{Mitrinovic}}{\ge}\frac{72.27Sr^2}{\frac{6\sum_{cyc}a^2}{R}+24\sqrt{3}sS}$$

$$=\frac{324Sr^2}{\frac{\sum_{cyc}a^2}{R}+4\sqrt{3}sS}\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Mitrinovic}}}{\ge}\frac{324Sr^2}{9R+18RS}=\frac{324Sr^2}{9R+20RS-2RS}\overset{\text{Euler}}{\ge}\frac{324Sr^2}{9R+20RS-4rS}$$

$$\Rightarrow\frac{6ar_a^2}{6+\sqrt{3}a(a+b+c)}+\frac{6br_b^2}{6+\sqrt{3}b(a+b+c)}+\frac{6cr_c^2}{6+\sqrt{3}c(a+b+c)}$$

$$\ge\frac{324Sr^2}{R(9+20S)-4Sr}\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$