ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $k^3 = h_a h_b h_c$ and $t^3 = (abc)^2$, the following relationship holds

$$\frac{r_a^2}{k+a^2}+\frac{r_b^2}{k+b^2}+\frac{r_c^2}{k+c^2} \geq \frac{108r^3}{t+6R^3}$$

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$$\sum_{cyc} a^2 r_a = rs\sum_{cyc}\frac{a^2}{s-a} = rs\sum_{cyc}\frac{(a-s+s)^2}{s-a}$$

$$= rs\left(\sum_{cyc}\frac{(s-a)^2}{s-a} - 2s\sum_{cyc}\frac{s-a}{s-a} + \frac{s^2}{r^2 s}\sum_{cyc}(s-b)(s-c)\right)$$

$$= rs\left(s-6s+\frac{s(4R+r)}{r}\right) \Rightarrow \sum_{cyc} a^2 r_a = 4(R-r)s^2 \to (1)$$

$$\text{Now, } \frac{\sqrt[3]{a^2b^2c^2}.(4R+r)^3}{(4R+r).324r^3.\sqrt[3]{h_a h_b h_c}} \overset{\text{Euler}}{\geq} \frac{\sqrt[3]{16R^2r^2s^2}.(9r)^2}{324r^3.\sqrt[3]{\frac{2r^2s^2}{R}}} = \frac{R}{2r} \overset{\text{Euler}}{\geq} 1$$

$$\Rightarrow t(4R+r)^3 \geq 324r^3k(4R+r) \to (2)$$

$$\text{Also, } \frac{R^3(4R+r)^3}{216(R-r)r^3s^2} \overset{\text{Mitrinovic}}{\geq} \frac{4R^3(4R+r)^3}{216(R-r)r^3.27R^2} \overset{?}{\geq} 1$$

$$\Leftrightarrow R(4R+r)^3 \overset{?}{\geq} 1458(R-r)r^3$$

$$\Leftrightarrow 64t^4+48t^3+12t^2-1457t+1458 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(64t^2+304t+972)+1215\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow 6R^3(4R+r)^3 \geq 108.12(R-r)r^3s^2 \to (3)$$

$$\text{So, } \frac{r_a^2}{k+a^2}+\frac{r_b^2}{k+b^2}+\frac{r_c^2}{k+c^2} = \frac{r_a^3}{kr_a+a^2r_a}+\frac{r_b^3}{kr_b+b^2r_b}+\frac{r_c^3}{kr_c+c^2r_c}$$

$$\overset{\substack{\text{Holder}\\ \text{and}\\ \text{via (1)}}}{\geq} \frac{(4R+r)^3}{3k(4R+r)+12(R-r)s^2} \overset{?}{\geq} \frac{108r^3}{t+6R^3}$$

$$\Leftrightarrow t(4R+r)^3+6R^3(4R+r)^3 \overset{?}{\geq} 324r^3k(4R+r)+108.12(R-r)r^3s^2$$

$$\to \text{true via (2) + (3)} \therefore \frac{r_a^2}{k+a^2}+\frac{r_b^2}{k+b^2}+\frac{r_c^2}{k+c^2} \geq \frac{108r^3}{t+6R^3}$$

$\forall \Delta$ ABC," = " iff Δ ABC is equilateral (QED)