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In ΔABC the following relationship holds:

$$\frac{bc\cdot r_a^2}{1+bc\cos(A)}+\frac{ca\cdot r_b^2}{1+ca\cos(B)}+\frac{ab\cdot r_c^2}{1+ab\cos(C)}\geq\frac{648r^4}{2+3R^2}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{bcr_a^2}{1+bc\cos(A)}+\frac{car_b^2}{1+ca\cos(B)}+\frac{abr_c^2}{1+ab\cos(C)}\overset{\text{divided by}}{=}\frac{r_a^2}{\frac{1}{bc}+\cos A}+\frac{r_b^2}{\frac{1}{ac}+\cos B}$$

$$+\frac{r_c^2}{\frac{1}{ab}+\cos C}\geq\frac{(r_a+r_b+r_c)^2}{\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}+\cos A+\cos B+\cos C\right)}=$$

$$=\frac{(r_a+r_b+r_c)^2}{\left(\frac{a+b+c}{abc}+1+\frac{r}{R}\right)}=\frac{(4R+r)^2}{\left(\frac{1}{2Rr}+1+\frac{r}{R}\right)}=\frac{(4R+r)^2}{\frac{1+2Rr+2r^2}{2Rr}}=\frac{(4R+r)^2\cdot 2Rr}{1+2Rr+2r^2}\overset{\text{Euler}}{\geq}$$

$$\overset{\text{Euler}}{\geq}\frac{(9r)^2\cdot 4r^2}{R^2+\frac{R^2}{2}+1}=\frac{648r^4}{2+3R^2}$$

Equality holds for $a=b=c$.