ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{bch_a{r_a}^2}{2bc+h_a}+\frac{ach_b{r_b}^2}{2ac+h_b}+\frac{abh_c{r_c}^2}{2ab+h_c}\ge\frac{324r^4}{4R+1}$$

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$$\frac{bch_a{r_a}^2}{2bc+h_a}+\frac{ach_b{r_b}^2}{2ac+h_b}+\frac{abh_c{r_c}^2}{2ab+h_c}=\frac{{r_a}^2}{\frac{2}{h_a}+\frac{1}{bc}}+\frac{{r_b}^2}{\frac{2}{h_b}+\frac{1}{ac}}+\frac{{r_c}^2}{\frac{2}{h_c}+\frac{1}{ab}}\ge$$

$$\ge\frac{(r_a+r_b+r_c)^2}{2\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)+\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)}=\frac{(r_a+r_b+r_c)^2}{\frac{2}{r}+\frac{a+b+c}{abc}}=\frac{(r_a+r_b+r_c)^2}{\frac{2}{r}+\frac{2P}{abc}}=$$

$$=\frac{(r_a+r_b+r_c)^2}{\frac{2}{r}+\frac{2S}{r\cdot 4RS}}=\frac{(r_a+r_b+r_c)^2}{\frac{2}{r}+\frac{1}{2Rr}}=\frac{(r_a+r_b+r_c)^2}{\frac{4R+1}{2Rr}}=\frac{2Rr(r_a+r_b+r_c)^2}{4R+1}\overset{Euler}{\ge}$$

$$\ge\frac{4r^2(r_a+r_b+r_c)^2}{4R+1}\quad(*)$$

$$\begin{cases}r_a+r_b=4Rcos^2\frac{C}{2}\\ r_b+r_c=4Rcos^2\frac{A}{2}\\ r_a+r_c=4Rcos^2\frac{B}{2}\end{cases}\Rightarrow r_a+r_b+r_c=2R\left(cos^2\frac{A}{2}+cos^2\frac{B}{2}+cos^2\frac{C}{2}\right)=$$

$$=R(1+cosA+1+cosB+1+cosC)=R(3+cosA+cosB+cosC)=$$

$$=R\left(3+\left(1+\frac{r}{R}\right)\right)=R\cdot\frac{4R+r}{R}=4R+r\quad(1)$$

$$(*)\overset{(1)}{\Rightarrow}\frac{4r^2(4R+r)^2}{4R+1}\overset{Euler}{\ge}\frac{4r^2(8r+r)^2}{4R+1}=\frac{324r^4}{4R+1}$$

Equality holds for $a=b=c$.