ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the $following$ $relationship$ $holds$:

$$\frac{\csc^2 A+\csc^2 B+\csc^2 C}{a+b+c}\ge\frac{R}{F}$$

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$$\csc^2 A+\csc^2 B+\csc^2 C=\sum_{cyc}\frac{4R^2}{a^2}=\frac{4R^2}{16R^2r^2s^2}\cdot\sum_{cyc}a^2b^2$$

$$=\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{4r^2s^2}\overset{?}{\ge}\frac{2sR}{rs}\Leftrightarrow(s^2+4Rr+r^2)^2-16Rrs^2\overset{?}{\ge}8Rrs^2$$

$$\Leftrightarrow s^4-(16Rr-2r^2)s^2+r^2(4R+r)^2\underset{(*)}{\overset{?}{\ge}}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\ge}(16Rr-5r^2)s^2-(16Rr-2r^2)s^2+r^2(4R+r)^2$$

$$=r^2((4R+r)^2-3s^2)\ge 0\text{ via Trucht (Doucet)}\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{\csc^2 A+\csc^2 B+\csc^2 C}{a+b+c}\ge\frac{R}{F}\ \forall\,\Delta\,ABC,''=''\ \text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$