ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{1}{r_a(r_a+r_b)}+\frac{1}{r_b(r_b+r_c)}+\frac{1}{r_c(r_a+r_c)}\geq\frac{2}{3R^2}$$

Proposed by Ertan Yildirim-Turkiye

Solution by Tapas Das-India

$$(r_a+r_b)(r_b+r_c)(r_c+r_a)=$$

$$=\left(\sum r_a\right)\left(\sum r_ar_b\right)-r_ar_br_c=(4R+r)s^2-s^2r=4Rs^2$$

$$\frac{1}{r_a(r_a+r_b)}\cdot\frac{1}{r_b(r_b+r_c)}\cdot\frac{1}{r_c(r_a+r_c)}=$$

$$=\frac{1}{r_ar_br_c(r_a+r_b)(r_b+r_c)(r_c+r_a)}=\frac{1}{s^2r.4Rs^2}\overset{Euler\,\&\,Mitrinovic}{\geq}$$

$$\geq\frac{1}{\frac{27}{4}R^2.\frac{R}{2}.4.R.\frac{27}{4}R^2}=\frac{8}{R^6.3^6}\ (1)$$

$$,\frac{1}{r_a(r_a+r_b)}+\frac{1}{r_b(r_b+r_c)}+\frac{1}{r_c(r_a+r_c)}\overset{Am-Gm}{\geq}$$

$$\geq 3\sqrt[3]{\frac{1}{r_a(r_a+r_b)}\cdot\frac{1}{r_b(r_b+r_c)}\cdot\frac{1}{r_c(r_a+r_c)}}\overset{(1)}{\geq}3\sqrt[3]{\frac{8}{R^6.3^6}}=\frac{2}{3R^2}$$

$$Equality\ holds\ for\ a=b=c$$