ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{9R}{2r} \le \sum_{cyc}\frac{r_a}{r_b}\sum_{cyc}\frac{r_b}{r_a} \le \frac{8R^2}{r^2} - 23$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2} \text{ and analogs}$$

$$\text{Now, } \sum_{cyc}\frac{r_a}{r_b}\sum_{cyc}\frac{r_b}{r_a} = 3 + \sum_{cyc}\frac{r_br_c}{r_a^2} + \sum_{cyc}\frac{r_a^2}{r_br_c}$$

$$= 3 + \frac{1}{r^2s}\sum_{cyc}(s-a)^3 + \sum_{cyc}\frac{(s-b)(s-c)}{(s-a)^2} \overset{A-G}{\le} 3 +$$

$$\frac{1}{r^2s}\left(\left(\sum_{cyc}(s-a)\right)^3 - 3\big((s-a)+(s-b)\big)\big((s-b)+(s-c)\big)\big((s-c)+(s-a)\big)\right)$$

$$+\frac{1}{4}\sum_{cyc}\frac{(a-s+s)^2}{(s-a)^2} = 3 + \frac{1}{r^2s}(s^3 - 3.4Rrs) + \frac{1}{4}\sum_{cyc}\frac{(s-a)^2 - 2s(s-a) + s^2}{(s-a)^2}$$

$$= 3 + \frac{s^2 - 12Rr}{r^2} + \frac{1}{4}\left(3 - \frac{2}{r}\sum_{cyc}r_a + \frac{1}{r^2}\sum_{cyc}r_a^2\right)$$

$$= 3 + \frac{s^2 - 12Rr}{r^2} + \frac{1}{4}\left(3 - \frac{2(4R+r)}{r} + \frac{(4R+r)^2 - 2s^2}{r^2}\right)$$

$$= 3 + \frac{s^2 - 12Rr}{r^2} + \frac{8R^2 + r^2 - s^2}{2r^2} \overset{?}{\le} \frac{8R^2}{r^2} - 23 \Leftrightarrow s^2 \overset{?}{\le} 8R^2 + 24Rr - 53r^2$$

$$\text{and} \because s^2 \overset{\text{Gerretsen}}{\le} 4R^2 + 4Rr + 3r^2 \therefore \text{it suffices to prove : } 4R^2 + 4Rr + 3r^2$$

$$\overset{?}{\le} 8R^2 + 24Rr - 53r^2 \Leftrightarrow R^2 + 5Rr - 14r^2 \overset{?}{\ge} 0 \Leftrightarrow (R-2r)(R+7r) \overset{?}{\ge} 0 \to \text{true}$$

$$\because R \overset{\text{Euler}}{\ge} 2r \therefore \boxed{\sum_{cyc}\frac{r_a}{r_b}\sum_{cyc}\frac{r_b}{r_a} \le \sum_{cyc}r_br_c(r_b+r_c) \le \frac{8R^2}{r^2} - 23}$$

$$\text{Again, } \sum_{cyc}\frac{r_a}{r_b}\sum_{cyc}\frac{r_b}{r_a} = 3 + \sum_{cyc}\frac{r_br_c}{r_a^2} + \sum_{cyc}\frac{r_a^2}{r_br_c} = 3$$

$$+\frac{1}{r^2s^4}\left(\left(\sum_{cyc}r_br_c\right)^3 - 3(r_br_c + r_cr_a)(r_cr_a + r_ar_b)(r_ar_b + r_br_c)\right)$$

$$+\frac{1}{r^2 s}\left(\begin{array}{c}\left(\sum_{cyc}(s-a)\right)^3 \\ -3\big((s-a)+(s-b)\big)\big((s-b)+(s-c)\big)\big((s-c)+(s-a)\big)\end{array}\right)\overset{via\ (i)}{=}$$

$$3+\frac{1}{r^2 s^4}\left(s^4-3rs^2.64R^3.\frac{s^2}{16R^2}\right)+\frac{1}{r^2 s}\left(s^3-3.4Rrs\right)\overset{?}{\geq}\frac{9R}{2r}$$

$$\Leftrightarrow 2s^4-(57Rr-6r^2)s^2+2r(4R+r)^3 \underbrace{\overset{?}{\geq}}_{(*)} 0 \text{ and } \because 2\left(s^2-16Rr+5r^2\right)^2 \overset{Gerretsen}{\geq} 0$$

$$\therefore \text{ in order to prove } (*), \text{it suffices to prove : LHS of } (*) \geq 2\left(s^2-16Rr+5r^2\right)^2$$

$$\Leftrightarrow 128R^3-416R^2r+344Rr^2-48r^3+7(R-2r)s^2 \overset{(**)}{\geq} 0$$

$$\text{Again, LHS of } (**) \overset{Gerretsen}{\geq} 128R^3-416R^2r+344Rr^2-48r^3$$

$$+7(R-2r)\left(16Rr-5r^2\right)\overset{?}{\geq}0 \Leftrightarrow 128t^3-304t^2+85t+22\overset{?}{\geq}0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big((t-2)(128t+208)+405\big)\overset{?}{\geq}0 \rightarrow \text{true} \because t \overset{Euler}{\geq} 2$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \boxed{\frac{9R}{2r} \leq \sum_{cyc}\frac{r_a}{r_b}\sum_{cyc}\frac{r_b}{r_a}} \text{ and hence,}$$

$$\frac{9R}{2r} \leq \sum_{cyc}\frac{r_a}{r_b}\sum_{cyc}\frac{r_b}{r_a} \leq \frac{8R^2}{r^2}-23\ \forall\ \Delta\ ABC, ''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$