ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$243r^4 \le w_a^3 r_a + w_b^3 r_b + w_c^3 r_c \le \left(\frac{3Rs}{2}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$w_a^3 r_a \overset{w_a \le \sqrt{s(s-a)}}{\le} s(s-a)\sqrt{s(s-a)}\frac{F}{s-a} = Fs\sqrt{s}\ \sqrt{s-a}$$

$$w_a^3 r_a + w_b^3 r_b + w_c^3 r_c = \sum w_a^3 r_a \le Fs\sqrt{s}\ \sum \sqrt{s-a} \overset{CBS}{\le}$$

$$\le\ rs^2\sqrt{s}\sqrt{3(s-a+s-b+s-c)} = rs^3\sqrt{3} \overset{Euler\ \&\ Mitrinovic}{\le} \frac{R}{2}s^2\frac{3\sqrt{3}}{2}R\sqrt{3} = \left(\frac{3Rs}{2}\right)^2$$

$$w_a w_b w_c \ge h_a h_b h_c \overset{GM-HM}{\ge} \left(\frac{3}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^2 = (3r)^3 = 27r^3$$

$$w_a^3 r_a + w_b^3 r_b + w_c^3 r_c \overset{AM-GM}{\ge} 3w_a w_b w_c \sqrt[3]{r_a r_b r_c} \ge$$

$$\ge 3.27r^3\sqrt[3]{s^2 r} \overset{Mitrinovic}{\ge} 81r^3\sqrt[3]{27r^3} = 243r^4$$

Equality holds for an equilateral triangle.