ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \cot^2\frac{A}{2} \geq \sqrt{3}\sum \cot\frac{A}{2}$$

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Solution by Tapas Das-India

$$\sum \cot^2\frac{A}{2} = \sum \frac{s^2}{r_a^2} = s^2\left(\left(\sum \frac{1}{r_a}\right)^2 - 2\sum \frac{1}{r_a r_b}\right) = s^2\left(\frac{1}{r^2} - 2\cdot\frac{4R+r}{s^2 r}\right) =$$

$$= \frac{s^2 - 8Rr - 2r^2}{r^2} \overset{Gerretsen}{\geq} \frac{16Rr - 5r^2 - 8Rr - 2r^2}{r^2} =$$

$$= \frac{8R}{r} - 7 = \frac{9R}{2r} + \frac{7R}{2r} - 7 \geq$$

$$\overset{Euler}{\geq} \frac{9R}{2r} + 7 - 7 = \frac{9R}{2r} \overset{Mitrinovic}{\geq} \frac{9}{2r}\frac{2s}{3\sqrt{3}} = \sqrt{3}\frac{s}{r} = \sqrt{3}\sum \cot\frac{A}{2}$$

Equality holds for an equilateral triangle.