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In any Δ ABC, the following relationship holds :

$$3\sqrt{3} \le \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right) \le 3\sqrt{3}\cdot\frac{R}{2r}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right) = \sum_{cyc}\left(\left(\sum_{cyc}\frac{1}{a}-\frac{1}{a}\right)r_a\right) = \left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}r_a\right) - \sum_{cyc}\frac{r_a}{a} =$$

$$= \frac{(s^2+4Rr+r^2)(4R+r)}{4Rrs} - \sum_{cyc}\frac{s\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}} =$$

$$= \frac{(s^2+4Rr+r^2)(4R+r)}{4Rrs} - \frac{s}{4R}.\frac{s^2+(4R+r)^2}{s^2} = \frac{4Rs^2}{4Rrs} \Rightarrow \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right)$$

$$= \frac{s}{r} \overset{\text{Mitrinovic}}{\le} 3\sqrt{3}.\frac{R}{2r} \text{ and also, } \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right) = \frac{s}{r} \overset{\text{Mitrinovic}}{\ge} 3\sqrt{3}$$

$$\text{and so, } 3\sqrt{3} \le \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right) \le 3\sqrt{3}.\frac{R}{2r}\ \forall\, \Delta\, ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)