# ROMANIAN MATHEMATICAL MAGAZINE

**In any $\Delta$ ABC, the following relationship holds :**

$$\sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)h_a\right)\le 3\sqrt{3}\le \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right)$$

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$$\sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right)=\sum_{cyc}\left(\left(\sum_{cyc}\frac{1}{a}-\frac{1}{a}\right)r_a\right)=\left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}r_a\right)-\sum_{cyc}\frac{r_a}{a}$$

$$=\frac{(s^2+4Rr+r^2)(4R+r)}{4Rrs}-\sum_{cyc}\frac{s\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}$$

$$=\frac{(s^2+4Rr+r^2)(4R+r)}{4Rrs}-\frac{s}{4R}\cdot\frac{s^2+(4R+r)^2}{s^2}=\frac{4Rs^2}{4Rrs}$$

$$\Rightarrow \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right)=\frac{s}{r}\to (1)$$

Now, $\sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)h_a\right)=\sum_{cyc}\left(\left(\frac{b+c}{bc}\right)\left(\frac{bc}{2R}\right)\right)=\frac{4s}{2R}\Rightarrow \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)h_a\right)=\frac{2s}{R}$

$\overset{\text{Mitrinovic}}{\le} 3\sqrt{3}\overset{\text{Mitrinovic}}{\le}\frac{s}{r}\overset{\text{via }(1)}{=}\frac{s}{r}$ and so, $\sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)h_a\right)\le 3\sqrt{3}$

$\le \sum_{cyc}\left(\left(\frac{1}{b}+\frac{1}{c}\right)r_a\right)\ \forall\ \Delta$ ABC, " = " iff $\Delta$ ABC is equilateral (QED)