ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{h_a}{\sqrt{9r^2+3h_a^2}}\le\frac{3}{2}$$

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$$\sum_{cyc}\frac{h_a}{\sqrt{9r^2+3h_a^2}}=\sum_{cyc}\frac{1}{\sqrt{\frac{9r^2a^2}{4r^2s^2}+3}}=\sum_{cyc}\frac{1}{\sqrt{\left(\frac{3a}{2s}\right)^2+3}}$$

$$\therefore\sum_{cyc}\frac{h_a}{\sqrt{9r^2+3h_a^2}}=\sum_{cyc}\frac{1}{\sqrt{x^2+3}}\left(x=\frac{3a}{2s},y=\frac{3b}{2s},z=\frac{3c}{2s}\right)\to(1)$$

$$\text{Now, }\frac{1}{\sqrt{x^2+3}}\overset{?}{\le}\frac{5-x}{8}\Leftrightarrow(x^2+3)(5-x)^2\overset{?}{\ge}64\left(\because x=\frac{3a}{2s}<\frac{3}{2}<5\right)$$

$$\Leftrightarrow x^4-10x^3+28x^2-30x+11\overset{?}{\ge}0\Leftrightarrow\frac{(x-1)^2}{4}.\left((2x-3)(2x-13)+5\right)\overset{?}{\ge}0$$

$$\to\text{true}\because x<\frac{3}{2}\Rightarrow(2x-3),(2x-13)<0\Rightarrow(2x-3)(2x-13)>0$$

$$\therefore\frac{1}{\sqrt{x^2+3}}\le\frac{5-x}{8}\text{ and analogs}\Rightarrow\sum_{cyc}\frac{1}{\sqrt{x^2+3}}\le\frac{1}{8}\left(15-\sum_{cyc}\frac{3a}{2s}\right)=\frac{12}{8}=\frac{3}{2}$$

$$\overset{\text{via }(1)}{\Rightarrow}\sum_{cyc}\frac{h_a}{\sqrt{9r^2+3h_a^2}}\le\frac{3}{2}\ \forall\,\Delta\,\text{ABC},''=''\ \text{iff }\Delta\,\text{ABC is equilateral (QED)}$$