ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $n \in \mathbb{N}^*$, the following relationship holds :

$$1+\sum_{cyc}\frac{\cot^{2n+1}\frac{A}{2}}{\cot^{2n-1}\frac{B}{2}} \geq 2\left(\frac{4R}{r}-3\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\cot^{2n+1}\frac{A}{2}}{\cot^{2n-1}\frac{B}{2}} = \sum_{cyc}\frac{\cot^{4n}\frac{A}{2}}{\cot^{2n-1}\frac{A}{2}\cdot\cot^{2n-1}\frac{B}{2}} = \sum_{cyc}\frac{\left(\cot^2\frac{A}{2}\right)^{2n}}{\left(\cot\frac{A}{2}\cot\frac{B}{2}\right)^{2n-1}} \overset{Radon}{\geq}$$

$$\frac{\left(\sum_{cyc}\cot^2\frac{A}{2}\right)^{2n-1}\cdot\left(\sum_{cyc}\cot^2\frac{A}{2}\right)}{\left(\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}\right)^{2n-1}} \geq \frac{\left(\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}\right)^{2n-1}\cdot\left(\sum_{cyc}\cot^2\frac{A}{2}\right)}{\left(\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}\right)^{2n-1}} = \sum_{cyc}\cot^2\frac{A}{2}$$

$$= s^2\sum_{cyc}\frac{1}{r_a^2} = \frac{s^2\left(s^4-2rs^2(4R+r)\right)}{r^2s^4} \Rightarrow 1+\sum_{cyc}\frac{\cot^{2n+1}\frac{A}{2}}{\cot^{2n-1}\frac{B}{2}} \geq \frac{s^2-8Rr-r^2}{r^2}$$

$$\overset{?}{\geq} 2\left(\frac{4R}{r}-3\right) \Leftrightarrow s^2-8Rr-r^2 \overset{?}{\geq} 8Rr-6r^2 \Leftrightarrow s^2 \overset{?}{\geq} 16Rr-5r^2$$

$\rightarrow$ **true via Gerretsen** $\therefore 1+\sum_{cyc}\frac{\cot^{2n+1}\frac{A}{2}}{\cot^{2n-1}\frac{B}{2}} \geq 2\left(\frac{4R}{r}-3\right)$ $\forall\,\Delta$ ABC **and**

$\forall\, n \in \mathbb{N}^*$, " = " **iff** Δ **ABC is equilateral (QED)**