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In ΔABC the following relationship holds:

$$\sum \frac{bc}{h_a} \le \sum \frac{bc}{r_a}$$

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$$In\ \Delta ABC\ wlog\ a \le b \le c \to h_a \ge h_b \ge h_c \to \begin{cases} \frac{1}{h_a} \le \frac{1}{h_b} \le \frac{1}{h_c} & (1) \\ bc \ge ac \ge ab & (2) \end{cases}$$

Let us consider conditions (1) *and* (2) *in the Chebyshev's inequality as well.*

$$\sum_{cyc} \frac{bc}{h_a} \le \frac{1}{3}(bc+ac+ab)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right) = \frac{1}{3}\left(\sum_{cyc} bc\right).\frac{1}{r} = \frac{1}{3}\left(\sum_{cyc} bc\right)\left(\sum_{cyc}\frac{1}{r_a}\right) \quad (*)$$

$$Again\ wlog\ a \le b \le c \to r_a \le r_b \le r_c \to \begin{cases} \frac{1}{r_a} \ge \frac{1}{r_b} \ge \frac{1}{r_c} & (3) \\ bc \ge ac \ge ab & (4) \end{cases}$$

According to Chebyshev's inequality:

$$\left(\sum_{cyc} bc\right)\left(\sum_{cyc}\frac{1}{r_a}\right) \le 3\sum \frac{bc}{r_a} \quad (**)$$

Let's use (**) *in* (*):

$$\sum \frac{bc}{h_a} \le 3.\frac{1}{3}\sum \frac{bc}{r_a} \to \sum \frac{bc}{h_a} \le \sum \frac{bc}{r_a} \quad (Proved)$$

Equality holds if triangle is an equilateral one.