ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $n \in \mathbb{N}^*$, the following relationship holds :

$$\sum_{cyc} \frac{\tan^{2n+1}\frac{A}{2}}{\tan^{2n-1}\frac{B}{2}} \ge 2\left(1 - \frac{r}{R}\right)$$

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$$\sum_{cyc} \frac{\tan^{2n+1}\frac{A}{2}}{\tan^{2n-1}\frac{B}{2}} = \sum_{cyc} \frac{\tan^{4n}\frac{A}{2}}{\tan^{2n-1}\frac{A}{2}.\tan^{2n-1}\frac{B}{2}} = \sum_{cyc} \frac{\left(\tan^2\frac{A}{2}\right)^{2n}}{\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)^{2n-1}} \overset{\text{Radon}}{\ge}$$

$$\frac{\left(\sum_{cyc}\tan^2\frac{A}{2}\right)^{2n-1}.\left(\sum_{cyc}\tan^2\frac{A}{2}\right)}{\left(\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2}\right)^{2n-1}} \ge \frac{\left(\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2}\right)^{2n-1}.\left(\sum_{cyc}\tan^2\frac{A}{2}\right)}{\left(\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2}\right)^{2n-1}}$$

$$= \sum_{cyc}\tan^2\frac{A}{2} = \frac{1}{s^2}\left((4R+r)^2 - 2s^2\right) = \frac{(4R+r)^2}{s^2} - 2 \overset{?}{\ge} 2\left(1 - \frac{r}{R}\right)$$

$$\Leftrightarrow R(4R+r)^2 \underset{(*)}{\overset{?}{\ge}} (4R-2r)s^2$$

$$\text{Now, RHS of } (*) \overset{\text{Rouche}}{\le} (4R-2r)\left(2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\le} R(4R+r)^2$$

$$\Leftrightarrow R(4R+r)^2 - (2R^2+10Rr-r^2)(4R-2r) \overset{?}{\ge} 2(4R-2r)(R-2r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow (R-2r)(8R^2-12Rr+r^2) \underset{(**)}{\overset{?}{\ge}} 2(4R-2r)(R-2r)\sqrt{R^2-2Rr}$$

$$\because R - 2r \overset{\text{Euler}}{\ge} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove :}$$

$$8R^2 - 12Rr + r^2 > 2(4R-2r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow \left(8R^2-12Rr+r^2\right)^2 - 4\left(R^2-2Rr\right)(4R-2r)^2 > 0$$

$$\Leftrightarrow r^2(4R+r)^2 > 0 \to \textit{true} \Rightarrow (**) \Rightarrow (*) \textit{ is true} \therefore \sum_{cyc} \frac{\tan^{2n+1}\frac{A}{2}}{\tan^{2n-1}\frac{B}{2}} \ge 2\left(1-\frac{r}{R}\right)$$

$\forall \Delta$ ABC and $\forall n \in \mathbb{N}^*$, " = " iff Δ ABC is equi$lateral$ (QED)