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In any Δ ABC, the following relationship holds :

$$\frac{4}{R} \le \sum_{cyc} \frac{h_b + h_c}{r_a^2} \le \frac{R}{r^2}$$

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$$\sum_{cyc} \frac{h_b + h_c}{r_a^2} = \frac{2rs\left(\frac{b+c}{bc}\right)}{\frac{r^2s^2}{(s-a)^2}} = \frac{2rs}{4Rrs.r^2s^2} \cdot \sum_{cyc}\left((s-a)^2\left(\sum_{cyc} ab - bc\right)\right)$$

$$= \frac{1}{2Rr^2s^2} \cdot \left((s^2+4Rr+r^2).\sum_{cyc}(s^2 - 2sa + a^2) - \sum_{cyc}(s^2bc - 2sabc + a^2bc)\right)$$

$$= \frac{(s^2+4Rr+r^2)\left(3s^2 - 4s^2 + 2(s^2 - 4Rr - r^2)\right) - s^2(s^2+4Rr+r^2) + 2s.12Rrs - 4Rrs.2s}{2Rr^2s^2}$$

$$= \frac{2r\left((4R-r)s^2 - r(4R+r)^2\right)}{2Rr^2s^2} \therefore \sum_{cyc} \frac{h_b + h_c}{r_a^2} \overset{(*)}{=} \frac{(4R-r)s^2 - r(4R+r)^2}{Rrs^2}$$

$$\therefore \sum_{cyc} \frac{h_b + h_c}{r_a^2} \le \frac{R}{r^2} \overset{\text{via }(*)}{\Leftrightarrow} \frac{(4R-r)s^2 - r(4R+r)^2}{Rrs^2} \le \frac{R}{r^2}$$

$$\Leftrightarrow (R^2 - 4Rr + r^2)s^2 + r^2(4R+r)^2 \overset{(1)}{\ge} 0$$

$\boxed{\textit{Case } 1}$ $R^2 - 4Rr + r^2 \ge 0$ *and* then, LHS of $(1) \ge r^2(4R+r)^2 > 0 \Rightarrow (1)$ is true (strict inequality)

$\boxed{\textit{Case } 2}$ $R^2 - 4Rr + r^2 < 0$ *and* then, LHS of $(1) = -\left(-(R^2 - 4Rr + r^2)\right)s^2$

$$+r^2(4R+r)^2 \overset{\text{Gerretsen}}{\ge} -\left(-(R^2 - 4Rr + r^2)\right)(4R^2 + 4Rr + 3r^2) + r^2(4R+r)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow 4t^4 - 12t^3 + 7t^2 + 4 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)\left((t-2)(4t^2 + 4t + 7) + 12\right) \overset{?}{\ge} 0$$

$\rightarrow$ true $\because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (1)$ is true $\therefore$ combining both cases, (1) is true $\forall\, \Delta$ ABC

$$\therefore \sum_{cyc} \frac{h_b + h_c}{r_a^2} \le \frac{R}{r^2} \forall\, \Delta \text{ ABC}$$

$$\text{Again, } \frac{4}{R} \le \sum_{cyc} \frac{h_b + h_c}{r_a^2} \overset{\text{via }(*)}{\Leftrightarrow} \frac{(4R-r)s^2 - r(4R+r)^2}{Rrs^2} \ge \frac{4}{R}$$

$$\Leftrightarrow (4R-5r)s^2 \overset{(2)}{\ge} r(4R+r)^2$$

$$\text{Now, } (4R-5r)s^2 \overset{\text{Gerretsen}}{\ge} (4R-5r)(16Rr - 5r^2) \overset{?}{\ge} r(4R+r)^2$$

$$\Leftrightarrow 48R^2 - 108Rr + 24r^2 \overset{?}{\ge} 0 \Leftrightarrow 12(4R-r)(R-2r) \overset{?}{\ge} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\Rightarrow (2) \text{ is true } \therefore \frac{4}{R} \le \sum_{cyc} \frac{h_b + h_c}{r_a^2} \text{ and so, } \frac{4}{R} \le \sum_{cyc} \frac{h_b + h_c}{r_a^2} \le \frac{R}{r^2}$$

$\forall \Delta ABC, '' = ''$ iff ΔABC is equilateral (QED)