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In any Δ ABC, the following relationship holds :

$$\frac{2}{r} \le \sum_{cyc} \frac{r_b + r_c}{h_a^2} \le \frac{R}{r^2}$$

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$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\sum_{cyc}\frac{r_b + r_c}{h_a^2} \le \frac{R}{r^2} \overset{\text{via (i) and analogs}}{\Leftrightarrow} \sum_{cyc}\frac{4R\cos^2\frac{A}{2}.a^2}{4r^2s^2} \le \frac{R}{r^2} \Leftrightarrow \sum_{cyc} a^2\cos^2\frac{A}{2} \le s^2$$

$$\Leftrightarrow \sum_{cyc}\left(a^3.\frac{s(s-a)}{abc}\right) \le s^2 \Leftrightarrow s\sum_{cyc} a^3 + 16r^2s^2 - 2\sum_{cyc} a^2b^2 \le 4Rrs^2$$

$$\Leftrightarrow s^2(s^2 - 6Rr - 3r^2) + 8r^2s^2 - (s^2 + 4Rr + r^2)^2 + 16Rrs^2 \le 2Rrs^2$$

$$\Leftrightarrow (2R + 3r)s^2 - r(4R + r)^2 \le 2Rs^2 \Leftrightarrow 3s^2 \le (4R + r)^2 \rightarrow \text{true via Trucht}$$

$$\therefore \sum_{cyc}\frac{r_b + r_c}{h_a^2} \le \frac{R}{r^2}$$

$$\text{Again, } \sum_{cyc}\frac{r_b + r_c}{h_a^2} \overset{\text{via (i) and analogs}}{=} \sum_{cyc}\frac{4R\left(\frac{1}{h_a}\right)^2}{\sec^2\frac{A}{2}} \overset{\text{Bergstrom}}{\ge} \frac{4R.\left(\sum_{cyc}\frac{1}{h_a}\right)^2}{\frac{s^2+(4R+r)^2}{s^2}}$$

$$= \frac{4Rs^2}{r^2(s^2 + (4R + r)^2)} \overset{?}{\ge} \frac{2}{r} \Leftrightarrow (2R - r)s^2 \underset{(*)}{\overset{?}{\ge}} r(4R + r)^2$$

$$\text{Now, } (2R - r)s^2 \overset{\text{Gerretsen}}{\ge} (2R - r)(16Rr - 5r^2) \overset{?}{\ge} r(4R + r)^2$$

$$\Leftrightarrow 8R^2 - 17Rr + 2r^2 \overset{?}{\ge} 0 \Leftrightarrow (R - r)(R - 2r) \overset{?}{\ge} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{r_b + r_c}{h_a^2} \ge \frac{2}{r} \therefore \frac{2}{r} \le \sum_{cyc}\frac{r_b + r_c}{h_a^2} \le \frac{R}{r^2}$$

$$\forall\, \Delta\, ABC, \text{"} = \text{" iff } \Delta\, ABC \text{ is equilateral (QED)}$$