ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$6 \le \sum_{cyc} \frac{h_b^2 + h_c^2}{r_a^2} \le 6\left(\frac{R}{2r}\right)$$

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$$\sum_{cyc} \frac{h_a^2}{r_a^2} = 4\sum_{cyc} \frac{s^2 - 2sa + a^2}{a^2} = \frac{4s^2 \sum_{cyc} a^2b^2}{16R^2r^2s^2} - \frac{8s\sum_{cyc} ab}{4Rrs} + 12$$

$$\Rightarrow \sum_{cyc} \frac{h_a^2}{r_a^2} = \frac{\sum_{cyc} a^2b^2 - 8Rr\sum_{cyc} ab + 48R^2r^2}{4R^2r^2} \to (1)$$

$$\sum_{cyc} \frac{h_b^2 + h_c^2}{r_a^2} = \sum_{cyc} \frac{\sum_{cyc} h_a^2 - h_a^2}{r_a^2} = \frac{\sum_{cyc} a^2b^2}{4R^2} \cdot \frac{\left(\sum_{cyc} r_br_c\right)^2 - 2r_ar_br_c\sum_{cyc} r_a}{r^2s^4} - \sum_{cyc} \frac{h_a^2}{r_a^2}$$

$$\overset{\text{via (1)}}{=} \frac{\sum_{cyc} a^2b^2}{4R^2} \cdot \frac{s^4 - 2rs^2(4R + r)}{r^2s^4} - \frac{\sum_{cyc} a^2b^2 - 8Rr\sum_{cyc} ab + 48R^2r^2}{4R^2r^2}$$

$$= \frac{-2r^2\sum_{cyc} a^2b^2 - 48R^2r^2s^2 + 8Rr\left(s^2\sum_{cyc} ab - \sum_{cyc} a^2b^2\right)}{4R^2r^2s^2}$$

$$\Rightarrow \sum_{cyc} \frac{h_b^2 + h_c^2}{r_a^2} = \frac{-s^4 + (24R^2 + 4Rr - 2r^2)s^2 - r(4R + r)^3}{2R^2s^2} \to \text{(i)}$$

$$\therefore \text{(i)} \Rightarrow \sum_{cyc} \frac{h_b^2 + h_c^2}{r_a^2} \ge 6 \Leftrightarrow \frac{-s^4 + (24R^2 + 4Rr - 2r^2)s^2 - r(4R + r)^3}{2R^2s^2} \ge 6$$

$$\Leftrightarrow -s^4 + (12R^2 + 4Rr - 2r^2)s^2 - r(4R + r)^3 \overset{(*)}{\ge} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\ge} -(4R^2 + 4Rr + 3r^2)s^2 + (12R^2 + 4Rr - 2r^2)s^2$$

$$-r(4R + r)^3 \overset{?}{\ge} 0 \Leftrightarrow (8R^2 - 5r^2)s^2 \underset{(**)}{\overset{?}{\ge}} r(4R + r)^3$$

$$\text{Again, } (8R^2 - 5r^2)s^2 \overset{\text{Gerretsen}}{\ge} (8R^2 - 5r^2)(16Rr - 5r^2) \overset{?}{\ge} r(4R + r)^3$$

$$\Leftrightarrow 16t^3 - 22t^2 - 23t + 6 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t - 2)(16t^2 + 10t - 3) \overset{?}{\ge} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \frac{h_b^2 + h_c^2}{r_a^2} \ge 6$$

$$\text{Also, (i)} \Rightarrow \sum_{cyc} \frac{h_b^2 + h_c^2}{r_a^2} \le 6\left(\frac{R}{2r}\right) \Leftrightarrow \frac{-s^4 + (24R^2 + 4Rr - 2r^2)s^2 - r(4R + r)^3}{2R^2s^2}$$

$$\le 6\left(\frac{R}{2r}\right) \Leftrightarrow rs^4 + (6R^3 - 24R^2r - 4Rr^2 + 2r^3)s^2 + r^2(4R + r)^3 \overset{(***)}{\ge} 0$$

$$\text{Now, LHS of } (***) \overset{\text{Gerretsen}}{\geq} r(16Rr - 5r^2)s^2 + (6R^3 - 24R^2r - 4Rr^2 + 2r^3)s^2$$

$$+r^2(4R + r)^3 \overset{?}{\geq} 0 \Leftrightarrow (6R^3 - 24R^2r + 12Rr^2 - 3r^3)s^2 + r^2(4R + r)^3 \underbrace{\overset{?}{\geq}}_{(****)} 0$$

$$\boxed{Case\ 1}\ 6R^3 - 24R^2r + 12Rr^2 - 3r^3 \geq 0 \text{ and then : LHS of } (****)$$

$$\geq r^2(4R + r)^3 > 0 \Rightarrow (****) \text{ is true (strict inequality)}$$

$$\boxed{Case\ 2}\ 6R^3 - 24R^2r + 12Rr^2 - 3r^3 < 0 \text{ and then : LHS of } (****)$$

$$= -\left(-(6R^3 - 24R^2r + 12Rr^2 - 3r^3)s^2\right) + r^2(4R + r)^3$$

$$\overset{\text{Gerretsen}}{\geq} -\left(-(6R^3 - 24R^2r + 12Rr^2 - 3r^3)(4R^2 + 4Rr + 3r^2)\right) + r^2(4R + r)^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow 12t^5 - 36t^4 + 17t^3 + 6t^2 + 18t - 4 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t - 2)\left((t - 2)(12t^3 + 12t^2 + 17t + 26) + 54\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****)$$

$$\text{is true} \therefore \text{combining both cases, } (****) \Rightarrow (***) \text{ is true } \forall \Delta ABC \therefore \sum_{cyc} \frac{h_b^2 + h_c^2}{r_a^2}$$

$$\leq 6\left(\frac{R}{2r}\right) \therefore 6 \leq \sum_{cyc} \frac{h_b^2 + h_c^2}{r_a^2} \leq 6\left(\frac{R}{2r}\right)\ \forall \Delta ABC, \text{"="} \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$