# ROMANIAN MATHEMATICAL MAGAZINE

**In any $\Delta$ ABC, the following relationship holds :**

$$6 \le \sum_{cyc} \frac{r_b^2 + r_c^2}{h_a^2} \le 6\left(\frac{R}{2r}\right)^3$$

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$$\sum_{cyc} \frac{r_a^2}{h_a^2} = \frac{1}{4}\sum_{cyc} \frac{(a-s+s)^2}{(s-a)^2} = \frac{1}{4}\left(3 - 2s\sum_{cyc}\frac{1}{s-a} + s^2\sum_{cyc}\frac{1}{(s-a)^2}\right)$$

$$= \frac{1}{4}\left(3 - \frac{2s(4Rr+r^2)}{r^2 s} + \frac{s^2}{r^4 s^2}\left(\left(\sum_{cyc}(s-b)(s-c)\right)^2 - 2(s-a)(s-b)(s-c)\sum_{cyc}(s-a)\right)\right)$$

$$= \frac{1}{4}\left(3 - \frac{2(4R+r)}{r} + \frac{r^2\left((4R+r)^2 - 2s^2\right)}{r^4}\right) \Rightarrow \sum_{cyc}\frac{r_a^2}{h_a^2} = \frac{8R^2 + r^2 - s^2}{2r^2} \rightarrow (1)$$

$$\therefore \sum_{cyc}\frac{r_b^2 + r_c^2}{h_a^2} = \sum_{cyc}\frac{\sum_{cyc} r_a^2 - r_a^2}{h_a^2} = \frac{\left((4R+r)^2 - 2s^2\right)\left(s^2 - 4Rr - r^2\right)}{2r^2 s^2} - \sum_{cyc}\frac{r_a^2}{h_a^2}$$

$$\overset{\text{via } (1)}{=} \frac{\left((4R+r)^2 - 2s^2\right)\left(s^2 - 4Rr - r^2\right)}{2r^2 s^2} - \frac{8R^2 + r^2 - s^2}{2r^2}$$

$$\Rightarrow \sum_{cyc}\frac{r_b^2 + r_c^2}{h_a^2} = \frac{-s^4 + \left(8R^2 + 16Rr + 2r^2\right)s^2 - r(4R+r)^3}{2r^2 s^2} \rightarrow (\mathrm{i})$$

$$\therefore (\mathrm{i}) \Rightarrow \sum_{cyc}\frac{r_b^2 + r_c^2}{h_a^2} \ge 6 \Leftrightarrow \frac{-s^4 + \left(8R^2 + 16Rr + 2r^2\right)s^2 - r(4R+r)^3}{2r^2 s^2} \ge 6$$

$$\Leftrightarrow -s^4 + \left(8R^2 + 16Rr - 10r^2\right)s^2 - r(4R+r)^3 \overset{(*)}{\ge} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\ge} -\left(4R^2 + 4Rr + 3r^2\right)s^2 + \left(8R^2 + 16Rr - 10r^2\right)s^2$$

$$-r(4R+r)^3 \overset{?}{\ge} 0 \Leftrightarrow \left(4R^2 + 12Rr - 13r^2\right)s^2 \underset{(**)}{\overset{?}{\ge}} r(4R+r)^3$$

$$\text{Again, } \left(4R^2 + 12Rr - 13r^2\right)s^2 \overset{\text{Gerretsen}}{\ge} \left(4R^2 + 12Rr - 13r^2\right)\left(\begin{matrix}16Rr \\ -5r^2\end{matrix}\right) \overset{?}{\ge} r(4R+r)^3$$

$$\Leftrightarrow 4r\left(31R^2 - 70Rr + 16r^2\right) \overset{?}{\ge} 0 \Leftrightarrow (31R - 8r)(R - 2r) \rightarrow \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \sum_{cyc}\frac{r_b^2 + r_c^2}{h_a^2} \ge 6$$

$$\text{Also, } (\mathrm{i}) \Rightarrow \sum_{cyc}\frac{r_b^2 + r_c^2}{h_a^2} \le 6\left(\frac{R}{2r}\right)^3 \Leftrightarrow \frac{-s^4 + \left(8R^2 + 16Rr + 2r^2\right)s^2 - r(4R+r)^3}{2r^2 s^2}$$

$$\le 6\left(\frac{R}{2r}\right)^3 \Leftrightarrow 2rs^4+\left(3R^3-16R^2r-32Rr^2-4r^3\right)s^2+2r^2(4R+r)^3 \overset{(***)}{\ge} 0$$

Now, LHS of $(***) \overset{\text{Gerretsen}}{\ge} 2r\left(16Rr-5r^2\right)s^2+\left(3R^3-16R^2r-32Rr^2-4r^3\right)s^2$

$$+2r^2(4R+r)^3 \overset{?}{\ge} 0 \Leftrightarrow \left(3R^3-16R^2r-14r^3\right)s^2+2r^2(4R+r)^3 \underbrace{\overset{?}{\ge}}_{(****)} 0$$

**Case 1** $3R^3-16R^2r-14r^3 \ge 0$ *and* then : LHS of $(****)$ $\ge 2r^2(4R+r)^3 > 0 \Rightarrow (****)$ is true (strict inequality)

**Case 2** $3R^3-16R^2r-14r^3 < 0$ *and* then : LHS of $(****)$

$$= -\left(-\left(3R^3-16R^2r-14r^3\right)s^2\right)+2r^2(4R+r)^3$$

$$\overset{\text{Gerretsen}}{\ge} -\left(-\left(3R^3-16R^2r-14r^3\right)\left(4R^2+4Rr+3r^2\right)\right)+2r^2(4R+r)^3 \overset{?}{\ge} 0$$

$$\Leftrightarrow 12t^5-52t^4+73t^3-8t^2-32t-40 \overset{?}{\ge} 0$$

$$\Leftrightarrow (t-2)\left((t-2)\left(12t^3-4t^2+9t+44\right)+108\right) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (****)$$

is true $\therefore$ combining both cases, $(****) \Rightarrow (***)$ is true $\forall\, \Delta\, ABC$

$$\therefore \sum_{cyc}\frac{r_b^2+r_c^2}{h_a^2} \le 6\left(\frac{R}{2r}\right)^3$$

$$\therefore 6 \le \sum_{cyc}\frac{r_b^2+r_c^2}{h_a^2} \le 6\left(\frac{R}{2r}\right)^3 \quad \forall\, \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$