ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}\le\sum_{cyc}\frac{r_b^2+r_c^2}{h_a^2}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a^2}{r_a^2}=4\sum_{cyc}\frac{s^2-2sa+a^2}{a^2}=\frac{4s^2\sum_{cyc}a^2b^2}{16R^2r^2s^2}-\frac{8s\sum_{cyc}ab}{4Rrs}+12$$

$$\Rightarrow\sum_{cyc}\frac{h_a^2}{r_a^2}=\frac{\sum_{cyc}a^2b^2-8Rr\sum_{cyc}ab+48R^2r^2}{4R^2r^2}\to(1)$$

$$\sum_{cyc}\frac{r_a^2}{h_a^2}=\frac{1}{4}\sum_{cyc}\frac{(a-s+s)^2}{(s-a)^2}=\frac{1}{4}\left(3-2s\sum_{cyc}\frac{1}{s-a}+s^2\sum_{cyc}\frac{1}{(s-a)^2}\right)$$

$$=\frac{1}{4}\left(3-\frac{2s(4Rr+r^2)}{r^2s}+\frac{s^2}{r^4s^2}\left(\left(\sum_{cyc}(s-b)(s-c)\right)^2-2(s-a)(s-b)(s-c)\sum_{cyc}(s-a)\right)\right)$$

$$=\frac{1}{4}\left(3-\frac{2(4R+r)}{r}+\frac{r^2\left((4R+r)^2-2s^2\right)}{r^4}\right)\Rightarrow\sum_{cyc}\frac{r_a^2}{h_a^2}=\frac{8R^2+r^2-s^2}{2r^2}\to(2)$$

$$\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}=\sum_{cyc}\frac{\sum_{cyc}h_a^2-h_a^2}{r_a^2}=\frac{\sum_{cyc}a^2b^2}{4R^2}\cdot\frac{\left(\sum_{cyc}r_br_c\right)^2-2r_ar_br_c\sum_{cyc}r_a}{r^2s^4}-\sum_{cyc}\frac{h_a^2}{r_a^2}$$

$$\overset{\text{via }(1)}{=}\frac{\sum_{cyc}a^2b^2}{4R^2}\cdot\frac{s^4-2rs^2(4R+r)}{r^2s^4}-\frac{\sum_{cyc}a^2b^2-8Rr\sum_{cyc}ab+48R^2r^2}{4R^2r^2}$$

$$=\frac{-2r^2\sum_{cyc}a^2b^2-48R^2r^2s^2+8Rr\left(s^2\sum_{cyc}ab-\sum_{cyc}a^2b^2\right)}{4R^2r^2s^2}$$

$$\Rightarrow\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}=\frac{-s^4+\left(24R^2+4Rr-2r^2\right)s^2-r(4R+r)^3}{2R^2s^2}\to(\text{i})$$

$$\sum_{cyc}\frac{r_b^2+r_c^2}{h_a^2}=\sum_{cyc}\frac{\sum_{cyc}r_a^2-r_a^2}{h_a^2}=\frac{\left((4R+r)^2-2s^2\right)\left(s^2-4Rr-r^2\right)}{2r^2s^2}-\sum_{cyc}\frac{r_a^2}{h_a^2}$$

$$\overset{\text{via }(2)}{=}\frac{\left((4R+r)^2-2s^2\right)\left(s^2-4Rr-r^2\right)}{2r^2s^2}-\frac{8R^2+r^2-s^2}{2r^2}$$

$$\Rightarrow \sum_{cyc} \frac{r_b^2 + r_c^2}{h_a^2} = \frac{-s^4 + (8R^2 + 16Rr + 2r^2)s^2 - r(4R + r)^3}{2r^2 s^2} \rightarrow \text{(ii)}$$

$$\therefore \text{(i), (ii)} \Rightarrow \sum_{cyc} \frac{h_b^2 + h_c^2}{r_a^2} \le \sum_{cyc} \frac{r_b^2 + r_c^2}{h_a^2}$$

$$\Leftrightarrow \frac{-s^4 + (24R^2 + 4Rr - 2r^2)s^2 - r(4R + r)^3}{2R^2 s^2} \le \frac{-s^4 + (8R^2 + 16Rr + 2r^2)s^2 - r(4R + r)^3}{2r^2 s^2}$$

$$\Leftrightarrow (8R^4 + 16R^3 r - 22R^2 r^2 - 4Rr^3 + 2r^4)s^2 - (R^2 - r^2)s^4$$
$$-r(64R^5 + 48R^4 r - 52R^3 r^2 - 47R^2 r^3 - 12Rr^4 - r^5) \overset{(*)}{\ge} 0$$

Now, LHS of $(*)$ $\overset{\text{Gerretsen}}{\ge} (8R^4 + 16R^3 r - 22R^2 r^2 - 4Rr^3 + 2r^4)s^2$

$$-(4R^2 + 4Rr + 3r^2)(R^2 - r^2)s^2$$
$$-r(64R^5 + 48R^4 r - 52R^3 r^2 - 47R^2 r^3 - 12Rr^4 - r^5)$$
$$= (4R^4 + 12R^3 r - 21R^2 r^2 + 5r^4)s^2$$
$$-r(64R^5 + 48R^4 r - 52R^3 r^2 - 47R^2 r^3 - 12Rr^4 - r^5)$$
$$\overset{\text{Gerretsen}}{\ge} (4R^4 + 12R^3 r - 21R^2 r^2 + 5r^4)(16Rr - 5r^2)$$
$$-r(64R^5 + 48R^4 r - 52R^3 r^2 - 47R^2 r^3 - 12Rr^4 - r^5) \overset{?}{\ge} 0$$
$$\Leftrightarrow 31t^4 - 86t^3 + 38t^2 + 23t - 6 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t - 2)\big((t - 2)(31t^2 + 38t + 66) + 135\big) \overset{?}{\ge} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{h_b^2 + h_c^2}{r_a^2} \le \sum_{cyc} \frac{r_b^2 + r_c^2}{h_a^2} \ \forall \Delta ABC, \text{"="} \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $h_a \le \sqrt{s(s-a)} = \sqrt{r_b r_c}$, then we have

$$\frac{{r_b}^2 + {r_c}^2}{{h_a}^2} = \frac{{h_a}^2({r_b}^2 + {r_c}^2)}{{h_a}^4} \ge \frac{{h_a}^2({r_b}^2 + {r_c}^2)}{{r_b}^2 {r_c}^2} = \frac{{h_a}^2}{{r_b}^2} + \frac{{h_a}^2}{{r_c}^2} \quad \text{(and analogs)}$$

Therefore

$$\sum_{cyc} \frac{{r_b}^2 + {r_c}^2}{{h_a}^2} \ge \sum_{cyc} \left(\frac{{h_a}^2}{{r_b}^2} + \frac{{h_a}^2}{{r_c}^2}\right) = \sum_{cyc} \left(\frac{{h_c}^2}{{r_a}^2} + \frac{{h_b}^2}{{r_a}^2}\right) = \sum_{cyc} \frac{{h_b}^2 + {h_c}^2}{{r_a}^2},$$

as desired. Equality holds iff ΔABC is equilateral.