ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the $following$ $relationship$ holds :

$$\frac{27r}{2Rp} \le \sum_{cyc} \frac{\sin^3 B + \sin^3 C}{h_a} \le \frac{27R}{8rp}$$

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For own convenience, $s \equiv p$

$$\sum_{cyc} \frac{\sin^3 B + \sin^3 C}{h_a} = \frac{1}{16R^3rs} \cdot \sum_{cyc} a(b^3 + c^3) = \frac{1}{16R^3rs} \cdot \sum_{cyc} \left(ab\left(\sum_{cyc} a^2 - a^2\right)\right)$$

$$= \frac{1}{16R^3rs} \cdot \left(\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right) - abc\left(\sum_{cyc} a\right)\right) = \frac{s^4 - r^2(4R+r)^2 - 4Rrs^2}{8R^3rs}$$

$$\overset{\text{Gerretsen}}{\le} \frac{s^2(4R^2+4Rr+3r^2) - r^2(4R+r)^2 - 4Rrs^2}{8R^3rs} \overset{\text{Gerretsen}}{\le}$$

$$\frac{(4R^2+3r^2)(4R^2+4Rr+3r^2) - r^2(4R+r)^2}{8R^3rs} \overset{?}{\le} \frac{27R}{8rs}$$

$$\Leftrightarrow 11t^4 - 16t^3 - 8t^2 - 4t - 8 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(11t^3 + 6t^2 + 4t + 4) \overset{?}{\ge} 0$$

$$\therefore \sum_{cyc} \frac{\sin^3 B + \sin^3 C}{h_a} \le \frac{27R}{8rp}$$

Again, $\sum_{cyc} \frac{\sin^3 B + \sin^3 C}{h_a} = \frac{2R}{8R^3} \cdot \sum_{cyc} \frac{b^3 + c^3}{bc} \ge \frac{1}{4R^2} \sum_{cyc} \frac{bc(b+c)}{bc} = \frac{4s}{4R^2} = \frac{2s^2}{2R^2s}$

$$\overset{\text{Gerretsen + Euler}}{\ge} \frac{27Rr}{2R^2s} = \frac{27r}{2Rs} \therefore \sum_{cyc} \frac{\sin^3 B + \sin^3 C}{h_a} \ge \frac{27r}{2Rp}$$

$$\therefore \frac{27r}{2Rp} \le \sum_{cyc} \frac{\sin^3 B + \sin^3 C}{h_a} \le \frac{27R}{8rp} \ \forall \, \Delta \, ABC, '' ='' \text{ iff } \Delta \, ABC \text{ is } equilateral \text{ (QED)}$$