ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{27r}{2Rp} \le \sum_{cyc} \frac{\sin^3 B + \sin^3 C}{r_a} \le \frac{27R^2}{16r^2p}$$

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For own convenience, $s \equiv p$

$$\sum_{cyc} a^4 = 2\sum_{cyc} a^2b^2 - 16r^2s^2 = 2(s^2+4Rr+r^2)^2 - 32Rrs^2 - 16r^2s^2$$

$$= 2s^4 - (16Rr+12r^2)s^2 + 2r^2(4R+r)^2 \overset{\text{Gerretsen}}{\le} (8R^2+8Rr+6r^2)s^2$$

$$-(16Rr+12r^2)s^2 + 2r^2(4R+r)^2 = (8R^2-8Rr-6r^2)s^2 + 2r^2(4R+r)^2$$

$$\overset{\text{Gerretsen}}{\le} (8R^2-8Rr-6r^2)(4R^2+4Rr+3r^2) + 2r^2(4R+r)^2 \overset{?}{\le} 54R^3(R-r)$$

$$\Leftrightarrow 11t^4 - 27t^3 + 16t + 8 \overset{?}{\ge} 0 \quad \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(11t^2+17t+24)+44\right) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow \sum_{cyc} a^4 \le 54R^3(R-r) \text{ and } \sum_{cyc} \frac{\sin^3 B + \sin^3 C}{r_a} = \frac{1}{8R^3rs}\cdot\sum_{cyc}\left((b^3+c^3)(s-a)\right)$$

$$= \frac{1}{8R^3rs}\cdot\sum_{cyc}\left(\left(\sum_{cyc} a^3 - a^3\right)(s-a)\right)$$

$$= \frac{1}{8R^3rs}\cdot\left(\left(\sum_{cyc} a^3\right)\left(\sum_{cyc}(s-a)\right) - s\sum_{cyc} a^3 + \sum_{cyc} a^4\right) \le \frac{54R^3(R-r)}{8R^3rs} \overset{?}{\le} \frac{27R^2}{16r^2s}$$

$$\Leftrightarrow R^2 \overset{?}{\ge} 4r(R-r) \Leftrightarrow (R-2r)^2 \overset{?}{\ge} 0 \to \text{true} \therefore \sum_{cyc} \frac{\sin^3 B + \sin^3 C}{r_a} \le \frac{27R^2}{16r^2p}$$

$$\text{Again, } \sum_{cyc} \frac{\sin^3 B + \sin^3 C}{r_a} = \frac{1}{8R^3rs}\cdot\sum_{cyc}\left((b^3+c^3)(s-a)\right)$$

$$= \frac{1}{8R^3rs}\cdot\sum_{cyc}\left(\left(\sum_{cyc} a^3 - a^3\right)(s-a)\right)$$

$$= \frac{1}{8R^3rs}\cdot\left(\left(\sum_{cyc} a^3\right)\left(\sum_{cyc}(s-a)\right) - s\sum_{cyc} a^3 + \sum_{cyc} a^4\right) = \frac{1}{8R^3rs}\cdot\sum_{cyc} a^4$$

$$\geq \frac{1}{8R^3rs}\cdot\frac{1}{3}\left(\sum_{cyc} a^2\right)^2 \geq \frac{1}{8R^3rs}\cdot\frac{1}{3}\left(\frac{1}{3}\left(\sum_{cyc} a\right)^2\right)^2 = \frac{1}{8R^3rs}\cdot\frac{1}{3}\left(\frac{4s^2}{3}\right)^2$$

$$\overset{\text{Gerretsen + Euler}}{\geq} \frac{1}{8R^3rs}\cdot\frac{1}{3}\left(\frac{2.27Rr}{3}\right)^2 = \frac{108R^2r^2}{8R^3rs} = \frac{27r}{2Rs} \therefore \sum_{cyc}\frac{\sin^3 B + \sin^3 C}{r_a} \geq \frac{27r}{2Rp}$$

$$\therefore \frac{27r}{2Rp} \leq \sum_{cyc}\frac{\sin^3 B + \sin^3 C}{r_a} \leq \frac{27R^2}{16r^2p}\ \forall\,\Delta\,ABC,'' ='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$