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In any Δ ABC, the following relationship holds :

$$\frac{9r}{4Rp} \le \sum_{cyc} \frac{\cos A}{b+c} \le \frac{9}{8p}$$

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For own convenience, $p \equiv s$

$$\text{Now, } \cos B + \cos C = \frac{c^2 + a^2 - b^2}{2ca} + \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{bc^2 + a^2 b - b^3 + ca^2 + b^2 c - c^3}{2abc}$$

$$= \frac{bc(b+c) - (b+c)(b^2 - bc + c^2) + a^2(b+c)}{2abc} \therefore \sum_{cyc} \frac{\cos A}{b+c}$$

$$= \left(\sum_{cyc} \cos A\right)\left(\sum_{cyc} \frac{1}{b+c}\right) - \sum_{cyc} \frac{bc(b+c) - (b+c)(b^2 - bc + c^2) + a^2(b+c)}{2abc(b+c)}$$

$$= \frac{R+r}{R} \cdot \frac{\left(2\sum_{cyc} ab + \sum_{cyc} a^2\right) + \sum_{cyc} ab}{2s(s^2 + 2Rr + r^2)} - \frac{1}{8Rrs}\left(2\sum_{cyc} ab - \sum_{cyc} a^2\right)$$

$$= \frac{R+r}{R} \cdot \frac{4s^2 + s^2 + 4Rr + r^2}{2s(s^2 + 2Rr + r^2)} - \frac{2(s^2 + 4Rr + r^2) - 2(s^2 - 4Rr - r^2)}{8Rrs}$$

$$\Rightarrow \sum_{cyc} \frac{\cos A}{b+c} \overset{(\bullet)}{=} \frac{(R+4r)s^2 - Rr(4R+r)}{2Rs(s^2 + 2Rr + r^2)}$$

$$\therefore \sum_{cyc} \frac{\cos A}{b+c} \le \frac{9}{8s} \overset{\text{via }(\bullet)}{\Leftrightarrow} \frac{(R+4r)s^2 - Rr(4R+r)}{2Rs(s^2 + 2Rr + r^2)} \le \frac{9}{8s}$$

$$\Leftrightarrow (5R - 16r)s^2 + 9R(2Rr + r^2) + 4Rr(4R + r) \overset{(*)}{\ge} 0$$

Case 1: $5R - 16r \ge 0$ and then : LHS of $(*) \ge 9R(2Rr + r^2) + 4Rr(4R + r) > 0$

$\Rightarrow (*)$ is true (strict inequality)

Case 2: $5R - 16r < 0$ and then : LHS of $(*)$

$$= -(16r - 5R)s^2 + 9R(2Rr + r^2) + 4Rr(4R + r) \overset{\text{Gerretsen}}{\ge}$$

$$-(16r - 5R)(4R^2 + 4Rr + 3r^2) + 9R(2Rr + r^2) + 4Rr(4R + r) \overset{?}{\ge} 0$$

$$\Leftrightarrow 10t^3 - 5t^2 - 18t - 24 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(10t^2 + 15t + 12) \overset{?}{\ge} 0 \to \text{true}$$

$\because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*)$ is true $\therefore$ combining cases 1 and 2, $(*)$ is true $\forall \Delta$ ABC

$$\therefore \text{in any } \Delta \text{ ABC}, \sum_{cyc} \frac{\cos A}{b+c} \le \frac{9}{8p}$$

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$$\text{Also, } \frac{9r}{4Rp} \le \sum_{cyc} \frac{\cos A}{b+c} \overset{\text{via } (\bullet)}{\Leftrightarrow} \frac{(R+4r)s^2 - Rr(4R+r)}{2Rs(s^2+2Rr+r^2)} \ge \frac{9r}{4Rs}$$

$$\Leftrightarrow (2R-r)s^2 - 9r(2Rr+r^2) - 2Rr(4R+r) \overset{(**)}{\ge} 0$$

$$\text{Again, LHS of } (**) \overset{\text{Gerretsen}}{\ge} (2R-r)(16Rr-5r^2) - 9r(2Rr+r^2) - 2Rr(4R+r)$$

$$\overset{?}{\ge} 0 \Leftrightarrow 12R^2 - 23Rr - 2r^2 \overset{?}{\ge} 0 \Leftrightarrow (12R+r)(R-2r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\Rightarrow (**) \text{ is true } \forall \Delta ABC \therefore \text{ in any } \Delta ABC, \sum_{cyc} \frac{\cos A}{b+c} \ge \frac{9r}{4Rp} \therefore \text{ in any } \Delta ABC,$$

$$\frac{9r}{4Rp} \le \sum_{cyc} \frac{\cos A}{b+c} \le \frac{9}{8p}, \text{ equalities iff } \Delta ABC \text{ is equilateral (QED)}$$