# ROMANIAN MATHEMATICAL MAGAZINE

In $any\ \Delta$ ABC, the $following$ $relationship$ $holds$ :

$$\frac{12r}{R} \le \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \le \frac{3R}{r}$$

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<u>Proof (1)</u>

$$\cos B+\cos C=\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab}=\frac{bc^2+a^2b-b^3+ca^2+b^2c-c^3}{2abc}$$

$$=\frac{bc(b+c)+a^2(b+c)-(b+c)(b^2-bc+c^2)}{2abc}=\frac{(b+c)(a^2-(b-c)^2)}{2abc}$$

$$=\frac{(b+c)(4(s-b)(s-c)(s-a))}{2abc(s-a)}=\frac{2r^2s}{abc}\cdot\frac{b+c}{s-a}\Rightarrow\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)$$

$$=\left(\frac{b^2+c^2}{bc}\right)\cdot\frac{2r^2s}{abc}\cdot\frac{b+c}{s-a}=\frac{2r^2s}{16R^2r^2s^2}\cdot\sum_{cyc}\frac{a(b+c)(b^2+c^2)}{s-a}$$

$$=\frac{1}{8R^2s}\cdot\sum_{cyc}\frac{a(s+s-a)(b^2+c^2)}{s-a}=\frac{1}{8R^2s}\cdot\left(\sum_{cyc}\frac{as(b^2+c^2)}{s-a}+\sum_{cyc}a(b^2+c^2)\right)$$

$$\overset{\text{Chebyshev}}{\le}\frac{1}{8R^2s}\cdot\left(\frac{s}{3}\left(\sum_{cyc}\frac{a}{s-a}\right)\sum_{cyc}(b^2+c^2)+\sum_{cyc}ab(2s-c)\right)$$

$$\because \text{WLOG assuming } a\ge b\ge c$$

$$\left(\Rightarrow\frac{a}{s-a}\ge\frac{b}{s-b}\ge\frac{c}{s-c}\text{ and } b^2+c^2\le c^2+a^2\le a^2+b^2\right)$$

$$=\frac{1}{8R^2s}\cdot\left(\frac{2s}{3}\left(\sum_{cyc}\frac{(a-s)+s}{s-a}\right)\left(\sum_{cyc}a^2\right)+2s(s^2+4Rr+r^2)-12Rrs\right)$$

$$\overset{\text{Leibnitz}}{\le}\frac{1}{8R^2s}\cdot\left(\frac{2s}{3}\left(-3+\frac{s(4Rr+r^2)}{r^2s}\right)(9R^2)+2s(s^2-2Rr+r^2)\right)$$

$$\overset{\text{Gerretsen}}{\le}\frac{1}{8R^2s}\cdot\left(\frac{2s(4R-2r).9R^2}{3r}+2s(4R^2+2Rr+4r^2)\right)$$

$$=\frac{9R^2(2R-r)+3r(2R^2+Rr+2r^2)}{6R^2r}\overset{?}{\le}\frac{3R}{r}$$

$$\Leftrightarrow 3r(R^2-Rr-2r^2)\overset{?}{\ge}0\Leftrightarrow 3r(R+r)(R-2r)\overset{?}{\ge}0\rightarrow\text{true}\because R\overset{\text{Euler}}{\ge}2r$$

$$\therefore\sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right)\le\frac{3R}{r}$$

$$\text{Again, }\sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right)\overset{A-G}{\ge}\sum_{cyc}(2(\cos B+\cos C))$$

$$\left(\because\cos B+\cos C=\frac{(b+c)(4(s-b)(s-c))}{2abc}>0\text{ and analogs}\right)$$

$$= 4\left(1+\frac{r}{R}\right) \overset{\text{Euler}}{\geq} 4\left(\frac{2r}{R}+\frac{r}{R}\right) \therefore \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \geq \frac{12r}{R}$$

$$\therefore \frac{12r}{R} \leq \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \leq \frac{3R}{r}$$

$\forall\, \Delta\, ABC,'' ='' \text{ iff } \Delta\, ABC$ is equilateral (QED)

**Proof (2)**

$$\sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \overset{\text{Bandila}}{\leq} \sum_{cyc}\left(\left(\frac{R}{r}\right)(\cos B+\cos C)\right)$$

$$\left(\because \cos B+\cos C = \frac{(b+c)\big(4(s-b)(s-c)\big)}{2abc} > 0\right) = \frac{2R}{r}\left(1+\frac{r}{R}\right) \overset{\text{Euler}}{\leq} \frac{2R}{r}\left(1+\frac{1}{2}\right) = \frac{3R}{r}$$

$$\text{Again}, \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \overset{A-G}{\geq} \sum_{cyc}\big(2(\cos B+\cos C)\big)$$

$$\left(\because \cos B+\cos C = \frac{(b+c)\big(4(s-b)(s-c)\big)}{2abc} > 0 \text{ and analogs}\right)$$

$$= 4\left(1+\frac{r}{R}\right) \overset{\text{Euler}}{\geq} 4\left(\frac{2r}{R}+\frac{r}{R}\right) = \frac{12r}{R} \therefore \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \geq \frac{12r}{R}$$

$$\therefore \frac{12r}{R} \leq \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \leq \frac{3R}{r}$$

$\forall\, \Delta\, ABC,'' ='' \text{ iff } \Delta\, ABC$ is equilateral (QED)