ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{3r}{R^2} \le \sum_{cyc} \frac{m_b + m_c}{b^2 + c^2} \le \frac{3R}{8r^2}$$

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$$\sum_{cyc} \frac{m_b + m_c}{b^2 + c^2} = \sum_{cyc} \frac{\sum_{cyc} m_a - m_a}{b^2 + c^2} = \left(\sum_{cyc} m_a\right)\left(\sum_{cyc} \frac{1}{b^2 + c^2}\right) - \sum_{cyc} \frac{m_a}{b^2 + c^2}$$

$$\overset{\substack{A-G,\\ \text{Tereshin} \\ \text{and} \\ \text{Leuenberger}}}{\le} (4R + r)\left(\frac{1}{2}\sum_{cyc} \frac{a}{bca}\right) - \frac{1}{4R}\sum_{cyc} \frac{b^2 + c^2}{b^2 + c^2} = \frac{(4R + r)s}{4Rrs} - \frac{3}{4R} = \frac{2R - r}{2Rr}$$

$$\overset{?}{\le} \frac{3R}{8r^2} \Leftrightarrow 3R^2 - 8Rr + 4r^2 \overset{?}{\ge} 0 \Leftrightarrow (3R - 2r)(R - 2r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\therefore \sum_{cyc} \frac{m_b + m_c}{b^2 + c^2} \le \frac{3R}{8r^2}$$

$$\text{Again}, \sum_{cyc} \frac{m_b + m_c}{b^2 + c^2} \overset{\text{Tereshin}}{\ge} \frac{1}{4R}\sum_{cyc} \frac{c^2 + a^2 + a^2 + b^2}{b^2 + c^2} = \frac{1}{4R}\left(3 + 2\sum_{cyc} \frac{a^2}{b^2 + c^2}\right)$$

$$\overset{\text{Nesbitt}}{\ge} \frac{3}{2R} = \frac{3r}{2Rr} \overset{\text{Euler}}{\ge} \frac{3r}{R^2} \therefore \sum_{cyc} \frac{m_b + m_c}{b^2 + c^2} \ge \frac{3r}{R^2}$$

$$\therefore \frac{3r}{R^2} \le \sum_{cyc} \frac{m_b + m_c}{b^2 + c^2} \le \frac{3R}{8r^2} \ \forall \Delta \text{ ABC}, '' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$