ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{m_a^2}{m_a+m_b}+\frac{m_b^2}{m_b+m_c}\ge\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right)$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right)=\frac{3F}{2a}+\frac{F}{b}-\frac{Rp}{4c}\overset{\text{Panaitopol}}{\le}\frac{3}{4}h_a+\frac{1}{2}h_b-\frac{m_c}{4}$$

$$\le\frac{3}{4}m_a+\frac{1}{2}m_b-\frac{m_c}{4}\therefore\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right)\le\frac{3m_a+2m_b-m_c}{4}\to(1)$$

$$\text{Again, }\frac{m_a^2}{m_a+m_b}+\frac{m_b^2}{m_b+m_c}\overset{\text{Bergstrom}}{\ge}\frac{(m_a+m_b)^2}{m_a+2m_b+m_c}\overset{?}{\ge}\frac{3m_a+2m_b-m_c}{4}$$

$$\Leftrightarrow 4(x+y)^2\overset{?}{\ge}(3x+2y-z)(x+2y+z)\ (x=m_a,y=m_b,z=m_c)$$

$$\Leftrightarrow 4x^2+4y^2+8xy\overset{?}{\ge}3x^2+6xy+3xz+2xy+4y^2+2yz-zx-2yz-z^2$$

$$\Leftrightarrow x^2-2xz+z^2\overset{?}{\ge}0\Leftrightarrow(x-z)^2\overset{?}{\ge}0\to\text{true}$$

$$\therefore\frac{m_a^2}{m_a+m_b}+\frac{m_b^2}{m_b+m_c}\ge\frac{3m_a+2m_b-m_c}{4}\overset{\text{via }(1)}{\ge}\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right)$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\frac{m_a{}^2}{m_a+m_b}+\frac{m_a+m_b}{4}\ge m_a\ \text{ and }\ \frac{m_b{}^2}{m_b+m_c}+\frac{m_b+m_c}{4}\ge m_b.$$

Adding these inequalities, we have

$$\frac{m_a{}^2}{m_a+m_b}+\frac{m_b{}^2}{m_b+m_c}\ge\frac{1}{4}(3m_a+2m_b-m_c).$$

Also, we have $m_a\ge h_a=\frac{2pr}{a}$, $m_b\ge h_b=\frac{2pr}{b}$ and $m_c\le\frac{Rh_c}{2r}=\frac{pR}{c}$.

Therefore

$$\frac{m_a{}^2}{m_a+m_b}+\frac{m_b{}^2}{m_b+m_c}\ge\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right).$$

Equality holds iff ΔABC is equilateral.