ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$9r \le \sum_{cyc} \frac{m_b^3 + m_c^3}{w_b^2 + w_c^2} \le \frac{9R^5}{32r^4}$$

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$$\text{WLOG if we assume } a \ge b \ge c, \text{then : } m_b^3 + m_c^3 \ge m_c^3 + m_a^3 \ge m_a^3 + m_b^3$$

$$\text{and } \frac{1}{w_b^2 + w_c^2} \le \frac{1}{w_c^2 + w_a^2} \le \frac{1}{w_a^2 + w_b^2} \therefore \sum_{cyc} \frac{m_b^3 + m_c^3}{w_b^2 + w_c^2} \overset{\text{Chebyshev}}{\le}$$

$$\frac{1}{3}\left(\sum_{cyc}(m_b^3 + m_c^3)\right)\left(\sum_{cyc} \frac{1}{w_b^2 + w_c^2}\right) \overset{A-G}{\le}$$

$$\frac{2}{3}\begin{pmatrix} m_a\left(\frac{2\sum_{cyc} a^2 - 3a^2}{4}\right) + m_b\left(\frac{2\sum_{cyc} a^2 - 3b^2}{4}\right) \\ +m_c\left(\frac{2\sum_{cyc} a^2 - 3c^2}{4}\right) \end{pmatrix}\left(\frac{1}{2}\sum_{cyc} \frac{1}{w_b w_c}\right)$$

$$\le \frac{1}{3}\left(\left(\frac{\sum_{cyc} a^2}{2}\right)\left(\sum_{cyc} m_a\right) - \frac{3}{4}\sum_{cyc} \frac{a^2}{\frac{1}{m_a}}\right)\left(\sum_{cyc} \frac{1}{w_a^2}\right) \overset{\substack{\text{Leuenberger + Euler} \\ and \\ \text{Bergstrom}}}{\le}$$

$$\frac{1}{3}\left((s^2 - 4Rr - r^2).\frac{9R}{2} - \frac{3}{4}.\frac{4s^2}{\sum_{cyc}\frac{1}{h_a}}\right)\left(\sum_{cyc} \frac{1}{h_a^2}\right)$$

$$= \frac{1}{3}\left((s^2 - 4Rr - r^2).\frac{9R}{2} - 3rs^2\right)\left(\frac{\sum_{cyc} a^2}{4r^2s^2}\right)$$

$$\overset{\text{Leibnitz}}{\le} \frac{(3R - 2r)s^2 - 3R(4Rr + r^2)}{2}.\frac{9R^2}{4r^2s^2} \overset{?}{\le} \frac{9R^5}{32r^4}$$

$$\Leftrightarrow (R^3 - 12Rr^2 + 8r^3)s^2 + r^3(48R^2 + 12Rr) \underbrace{\overset{?}{\ge}}_{(*)} 0$$

$\boxed{Case\ 1}$ $R^3 - 12Rr^2 + 8r^3 \ge 0$ *and* then : LHS of $(*) \ge r^3(48R^2 + 12Rr) > 0$

$\Rightarrow (*)$ is true (strict inequality)

$\boxed{Case\ 2}$ $R^3 - 12Rr^2 + 8r^3 < 0$ *and* then : LHS of $(*)$

$$= -\left(-(R^3 - 12Rr^2 + 8r^3)\right)s^2 + r^3(48R^2 + 12Rr) \overset{\text{Gerretsen}}{\ge}$$

$$-\left(-(R^3 - 12Rr^2 + 8r^3)\right)(4R^2 + 4Rr + 3r^2) + r^3(48R^2 + 12Rr) \overset{?}{\ge} 0$$

$$\Leftrightarrow 4t^5 + 4t^4 - 45t^3 + 32t^2 + 8t + 24 \overset{?}{\ge} 0 \ \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(4t^3 + 20t^2 + 19t + 28) + 44\right) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*)$$

is true and combining both $cases$, $(*)\ \forall\ \Delta\ ABC\ \therefore \sum_{cyc} \frac{m_b^3 + m_c^3}{w_b^2 + w_c^2} \le \frac{9R^5}{32r^4}\ \forall\ \Delta\ ABC$

$$Again, \sum_{cyc} \frac{m_b^3 + m_c^3}{w_b^2 + w_c^2} \ge \sum_{cyc} \frac{w_b^3 + w_c^3}{w_b^2 + w_c^2} \overset{\text{Chebyshev}}{\ge} \sum_{cyc} \frac{\frac{1}{2}(w_b^2 + w_c^2)(w_b + w_c)}{w_b^2 + w_c^2} = \sum_{cyc} w_a$$

$$\ge \sum_{cyc} h_a = \sum_{cyc} \frac{2rs}{a} \overset{\text{Bergstrom}}{\ge} \frac{2rs.9}{2s} \therefore \sum_{cyc} \frac{m_b^3 + m_c^3}{w_b^2 + w_c^2} \ge 9r$$

$$\therefore 9r \le \sum_{cyc} \frac{m_b^3 + m_c^3}{w_b^2 + w_c^2} \le \frac{9R^5}{32r^4}\ \forall\ \Delta\ ABC, '' ='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$