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In ΔABC holds :

$$9r \le \sum_{cyc} \frac{m_b^{\,3}+m_c^{\,3}}{w_b^{\,2}+w_c^{\,2}} \le \frac{9R^5}{32r^4}$$

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$$\sum_{cyc} \frac{m_b^{\,3}+m_c^{\,3}}{w_b^{\,2}+w_c^{\,2}} \overset{m_a \ge w_a}{\underbrace{\ge}} \sum_{cyc} \frac{w_b^{\,3}+w_c^{\,3}}{w_b^{\,2}+w_c^{\,2}} \overset{Chebyshev}{\underbrace{\ge}} \sum_{cyc} \frac{w_b+w_c}{2} \overset{w_a \ge h_a}{\underbrace{\ge}} \sum_{cyc} h_a \overset{AM-HM}{\underbrace{\ge}} \frac{9}{\sum_{cyc}\frac{1}{h_a}} = 9r.$$

$$\sum_{cyc} \frac{m_b^{\,3}+m_c^{\,3}}{w_b^{\,2}+w_c^{\,2}} \overset{\substack{Panaitopol\\AM-GM}}{\underbrace{\le}} \sum_{cyc} \frac{\frac{Rh_b}{2r}.m_b^{\,2}+\frac{Rh_c}{2r}.m_c^{\,2}}{2w_bw_c} \overset{w_a \le h_a}{\underbrace{\le}} \frac{R}{2r}\sum_{cyc} \frac{h_bm_b^{\,2}+h_cm_c^{\,2}}{2h_bh_c}$$

$$= \frac{R}{4r}\sum_{cyc}\left(\frac{m_b^{\,2}}{h_c}+\frac{m_c^{\,2}}{h_b}\right) = \frac{R}{4r}\sum_{cyc}\left(\frac{m_a^{\,2}}{h_b}+\frac{m_a^{\,2}}{h_c}\right) = \frac{R}{4r}\sum_{cyc}\frac{b+c}{2F}.\frac{2b^2+2c^2-a^2}{4}$$

$$= \frac{R}{32sr^2}\left(3\sum_{cyc}a^3+\sum_{cyc}a.\sum_{cyc}a^2\right) = \frac{R}{32sr^2}\left[3.2s\left(s^2-3r^2-6Rr\right)+2s.2\left(s^2-r^2-4Rr\right)\right]$$

$$= \frac{R\left(5s^2-11r^2-26Rr\right)}{16r^2} \overset{Gerretsen}{\underbrace{\le}} \frac{R\left(20R^2-4r^2-6Rr\right)}{16r^2} \overset{?}{\underbrace{\le}} \frac{9R^5}{32r^4}$$

$$\Leftrightarrow R(R-2r)\left(9R^3+18R^2r-4Rr^2+4r^3\right) \ge 0,$$

which is true by Euler's inequality.
So the proof is complete. Equality holds iff ΔABC is equilateral.