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In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\tan^4\frac{A}{2}+\tan^4\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}\geq 1$$

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Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{\tan^4\frac{A}{2}+\tan^4\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}=\sum_{cyc}\frac{\tan^4\frac{A}{2}+\tan^4\frac{B}{2}}{\tan^3\frac{A}{2}+\tan^3\frac{B}{2}}\cdot\frac{\tan^3\frac{A}{2}+\tan^3\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}\geq$$

$$\overset{LEHMER}{\geq}\sum_{cyc}\frac{\tan^3\frac{A}{2}+\tan^3\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}\cdot\frac{\tan^3\frac{A}{2}+\tan^3\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}=$$

$$=\sum_{cyc}\left(\frac{\tan^3\frac{A}{2}+\tan^3\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}\right)^2\overset{LEHMER}{\geq}\sum_{cyc}\left(\frac{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}\right)^2\geq$$

$$\overset{LEHMER}{\geq}\sum_{cyc}\left(\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1+1}\right)^2\overset{AM-GM}{\geq}\sum_{cyc}\left(\sqrt{\tan\frac{A}{2}\cdot\tan\frac{B}{2}}\right)^2=$$

$$=\sum_{cyc}\tan\frac{A}{2}\cdot\tan\frac{B}{2}=1$$

Equality holds for $A=B=C=\frac{\pi}{3}$.