ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{8R}{r} \leq \sum_{cyc} \frac{(b+c)^2}{r_a^2} \leq 16\left(\frac{R^2}{r^2} - 3\right)$$

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WLOG we may assume $a \geq b \geq c$ and then :

$$(b+c)^2 \leq (c+a)^2 \leq (a+b)^2 \text{ and } \frac{1}{r_a^2} \leq \frac{1}{r_b^2} \leq \frac{1}{r_c^2} \therefore \text{ via Chebyshev,}$$

$$\sum_{cyc} \frac{(b+c)^2}{r_a^2} \geq \left(\sum_{cyc}(b+c)^2\right)\left(\sum_{cyc} \frac{1}{r_a^2}\right)$$

$$\overset{A-G}{\geq} 4\left(\sum_{cyc} ab\right) . \frac{\left(\sum_{cyc} r_b r_c\right)^2 - 2r_a r_b r_c \sum_{cyc} r_a}{r^2 s^4}$$

$$= 4(s^2 + 4Rr + r^2) . \frac{s^4 - 2rs^2(4R+r)}{r^2 s^4} \overset{?}{\geq} \frac{8R}{r}$$

$$\Leftrightarrow s^4 - (10Rr + r^2)s^2 - 2r^2(4R+r)^2 \underset{(*)}{\overset{?}{\geq}} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} (6Rr - 6r^2)s^2 - 2r^2(4R+r)^2 \overset{\text{Gerretsen}}{\geq}$$

$$(6Rr - 6r^2)(16Rr - 5r^2) - 2r^2(4R+r)^2 \overset{?}{\geq} 0 \Leftrightarrow 32R^2 - 71Rr + 14r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R - 2r)(32R - 7r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (*) \text{ is true} \therefore \sum_{cyc} \frac{(b+c)^2}{r_a^2} \geq \frac{8R}{r}$$

$$\text{Again, } \sum_{cyc}\left((b+c)^2(s-a)^2\right) = \sum_{cyc}\left((s+s-a)^2(s-a)^2\right)$$

$$= \sum_{cyc}\left(\left(s^2 + (s-a)^2 + 2s(s-a)\right)(s-a)^2\right)$$

$$= s^2 \sum_{cyc}(s^2 - 2sa + a^2) + 2s\sum_{cyc}(s^3 - 3s^2 a + 3sa^2 - a^3)$$

$$+ \sum_{cyc}(s^4 - 4s^3 a + 6s^2 a^2 - 4sa^3 + a^4)$$

$$= s^2 . 3s^2 - 2s^3 . 2s + s^2 \sum_{cyc} a^2 + 2s . 3s^3 - 6s^3 . 2s + 6s^2 \sum_{cyc} a^2 - 2s \sum_{cyc} a^3 + 3s^4$$

$$-4s^3.2s+6s^2\sum_{cyc}a^2-4s\sum_{cyc}a^3+2\sum_{cyc}a^2b^2-16r^2s^2$$

$$=-12s^4+26s^2(s^2-4Rr-r^2)-12s^2(s^2-6Rr-3r^2)$$

$$+2\left(\left(s^2+4Rr+r^2\right)^2-16Rrs^2\right)-16r^2s^2$$

$$=2\left(2s^4-(24Rr+r^2)s^2+r^2(4R+r)^2\right)$$

$$\Rightarrow\sum_{cyc}\frac{(b+c)^2}{r_a^2}=\frac{2\left(2s^4-(24Rr+r^2)s^2+r^2(4R+r)^2\right)}{r^2s^2}\overset{?}{\le}16\left(\frac{R^2}{r^2}-3\right)$$

$$\Leftrightarrow 2s^4+r^2(4R+r)^2\underbrace{\overset{?}{\le}}_{(**)}(8R^2+24Rr-23r^2)s^2$$

$$\text{Now, LHS of }(**)\overset{\text{Gerretsen}}{\le}(8R^2+8Rr+6r^2)s^2+r^2(4R+r)^2\overset{?}{\le}$$

$$(8R^2+24Rr-23r^2)s^2\Leftrightarrow(16Rr-29r^2)s^2\underbrace{\overset{?}{\ge}}_{(***)}r^2(4R+r)^2$$

$$\text{Again, LHS of }(***)\overset{\text{Gerretsen}}{\ge}(16Rr-29r^2)(16Rr-5r^2)\overset{?}{\ge}r^2(4R+r)^2$$

$$\Leftrightarrow 240R^2-552Rr+144r^2\overset{?}{\ge}0\Leftrightarrow 24(R-2r)(10R-3r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r$$

$$\Rightarrow(***)\Rightarrow(**)\text{ is true}\therefore\sum_{cyc}\frac{(b+c)^2}{r_a^2}\le16\left(\frac{R^2}{r^2}-3\right)\therefore\frac{8R}{r}\le\sum_{cyc}\frac{(b+c)^2}{r_a^2}$$

$$\le16\left(\frac{R^2}{r^2}-3\right)\ \forall\Delta ABC,''='' \text{ iff }\Delta ABC\text{ is equilateral (QED)}$$