ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$12 \leq \sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} \leq \frac{3R^2}{r^2}$$

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$$\sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} = \sum_{cyc} \frac{(bc + ca)(bc + ab)}{ca.ab} = \sum_{cyc} \frac{(a+b)(c+a)}{a^2}$$

$$= \sum_{cyc} \frac{a^2 + \sum_{cyc} ab}{a^2} = 3 = 3 + \frac{1}{16R^2r^2s^2}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^2\right)$$

$$\geq 3 + \frac{1}{48R^2r^2s^2}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} ab\right)^2 \overset{\text{via }(\bullet)}{\geq} 3 + \frac{24Rrs^2}{48R^2r^2s^2}(s^2 + 4Rr + r^2)$$

$$= \frac{s^2 + 10Rr + r^2}{2Rr} \overset{\text{Gerretsen}}{\geq} \frac{16Rr - 5r^2 + 10Rr + r^2}{2Rr} = \frac{26Rr - 4r^2}{2Rr} \overset{\text{Euler}}{\geq} \frac{26Rr - 2Rr}{2Rr}$$

$$= 12 \sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} \geq 12$$

$$\text{Again}, \sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} = \sum_{cyc} \frac{(a+b)(c+a)}{a^2} \leq \sum_{cyc} \frac{(a+b)(c+a)}{4(s-b)(s-c)}$$

$$= \frac{2s(s^2 + 2Rr + r^2)}{4r^2s} \cdot \sum_{cyc} \frac{s-a}{(b+c)} = \frac{s^2 + 2Rr + r^2}{2r^2} \cdot \sum_{cyc} \frac{2s - a - s}{(b+c)}$$

$$= \frac{s^2 + 2Rr + r^2}{2r^2} \cdot \left(3 - s\sum_{cyc} \frac{1}{b+c}\right) \overset{\text{Bergstrom}}{\leq} \frac{s^2 + 2Rr + r^2}{2r^2} \cdot \left(3 - \frac{9s}{4s}\right) \overset{\text{Gerretsen}}{\leq}$$

$$\frac{4R^2 + 6Rr + 4r^2}{2r^2} \cdot \frac{3}{4} \overset{\text{Euler}}{\leq} \frac{4R^2 + 3R^2 + R^2}{2r^2} \cdot \frac{3}{4} \therefore \sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} \leq \frac{3R^2}{r^2}$$

$$\therefore 12 \leq \sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} \leq \frac{3R^2}{r^2}$$

$\forall \Delta$ ABC, $'' ='' $ iff Δ ABC is equilateral (QED)