ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c}\ge\left(\frac{2r}{R}\right)^3\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}ab\right)^2\overset{?}{\ge}24Rrs^2\Leftrightarrow\left(s^2+4Rr+r^2\right)^2\overset{?}{\ge}24Rrs^2$$

$$\Leftrightarrow s^4+2\left(4Rr+r^2\right)s^2+\left(4Rr+r^2\right)^2\overset{?}{\ge}24Rrs^2$$

$$\Leftrightarrow s^4-\left(16Rr-2r^2\right)s^2+\left(4Rr+r^2\right)^2\underset{(1)}{\overset{?}{\ge}}0$$

Now, LHS of (1) $\overset{\text{Gerretsen}}{\ge}\left(16Rr-5r^2\right)s^2-\left(16Rr-2r^2\right)s^2+\left(4Rr+r^2\right)^2$

$$=r^2\left((4R+r)^2-3s^2\right)\overset{\text{Trucht}}{\ge}0\Rightarrow(1)\text{ is true}\therefore\left(\sum_{cyc}ab\right)^2\overset{(\bullet)}{\ge}24Rrs^2$$

$$\text{Now, }\sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c}=\sum_{cyc}\frac{(bc+ca)(bc+ab)}{ca.ab}=\sum_{cyc}\frac{(a+b)(c+a)}{a^2}$$

$$=\sum_{cyc}\frac{a^2+\sum_{cyc}ab}{a^2}=3+\frac{1}{16R^2r^2s^2}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2b^2\right)$$

$$\ge3+\frac{1}{48R^2r^2s^2}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}ab\right)^2\overset{\text{via }(\bullet)}{\ge}3+\frac{24Rrs^2}{48R^2r^2s^2}\left(s^2+4Rr+r^2\right)$$

$$=\frac{s^2+10Rr+r^2}{2Rr}\overset{\text{Gerretsen}}{\ge}\frac{16Rr-5r^2+10Rr+r^2}{2Rr}$$

$$\Rightarrow\sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c}\overset{(\blacksquare)}{\ge}\frac{13R-2r}{R}$$

$$\text{Again, }\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}=\sum_{cyc}\frac{\left(\frac{rs}{s-a}+\frac{rs}{s-b}\right)\left(\frac{rs}{s-a}+\frac{rs}{s-c}\right)}{\frac{rs}{s-b}\cdot\frac{rs}{s-c}}$$

$$=\sum_{cyc}\frac{\frac{2s-a-b}{(s-a)(s-b)}\cdot\frac{2s-a-c}{(s-a)(s-c)}}{\frac{1}{(s-b)(s-c)}}=\sum_{cyc}\frac{bc}{(s-a)^2}\overset{A-G}{\le}\sum_{cyc}\frac{(b+c)^2}{4(s-a)^2}=\sum_{cyc}\frac{(s+s-a)^2}{4(s-a)^2}$$

$$=\sum_{cyc}\frac{s^2+(s-a)^2+2s(s-a)}{4(s-a)^2}=\frac{3}{4}+\frac{1}{2r}\sum_{cyc}\frac{rs}{s-a}+\frac{1}{4r^2}\sum_{cyc}\frac{r^2s^2}{(s-a)^2}$$

$$=\frac{3}{4}+\frac{1}{2r}\sum_{cyc} r_a+\frac{1}{4r^2}\sum_{cyc} r_a^2=\frac{3}{4}+\frac{4R+r}{2r}+\frac{(4R+r)^2-2s^2}{4r^2}$$

$$=\frac{(4R+r)^2-2s^2+2r(4R+r)+3r^2}{4r^2}$$

$$\overset{\text{Gerretsen}}{\leq}\frac{(4R+r)^2-2(16Rr-5r^2)+2r(4R+r)+3r^2}{4r^2}$$

$$\therefore\left(\frac{2r}{R}\right)^3\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}\overset{(\blacksquare\blacksquare)}{\leq}\frac{32r(R^2-Rr+r^2)}{R^3}\therefore(\blacksquare),(\blacksquare\blacksquare)\Rightarrow$$

$$\text{in order to prove : }\sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c}\geq\left(\frac{2r}{R}\right)^3\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c},$$

$$\text{it suffices to prove : }\frac{13R-2r}{R}\geq\frac{32r(R^2-Rr+r^2)}{R^3}$$

$$\Leftrightarrow 13t^3-34t^2+32t-32\geq 0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(9t^2+4t(t-2)+16)\geq 0$$

$$\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2\therefore\sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c}\geq\left(\frac{2r}{R}\right)^3\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}$$

$$\forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c}=\frac{(h_a+h_b)(h_b+h_c)(h_a+h_c)}{h_ah_bh_c}\cdot\sum_{cyc}\frac{h_a}{h_b+h_c}\underset{Nesbitt}{\overset{Cesaro}{\geq}}8.\frac{3}{2}=12.$$

$$\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}=\frac{4R}{r}.\sum_{cyc}\frac{r_a}{r_b+r_c}\overset{CBS}{\leq}\frac{4R}{r}.\sum_{cyc}\frac{r_a}{4}\left(\frac{1}{r_b}+\frac{1}{r_c}\right)$$

$$=\frac{R}{r}\left(\sum_{cyc}r_a\cdot\sum_{cyc}\frac{1}{r_a}-3\right)=\frac{R}{r}\left(\frac{4R+r}{r}-3\right)=$$

$$=\frac{3R^3}{2r^3}-\frac{R}{r}\left(\frac{R}{2r}-1\right)\left(\frac{3R}{r}-2\right)\overset{Euler}{\leq}\frac{3R^3}{2r^3}.$$

Therefore

$$\sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c}\geq 12=\left(\frac{2r}{R}\right)^3\cdot\frac{3R^3}{2r^3}\geq\left(\frac{2r}{R}\right)^3\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}$$

Equality holds iff ΔABC is equilateral.