ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta\, ABC$, the following relationship holds :

$$\frac{6R}{r} \le \sum_{cyc} \frac{(r_a + r_b)(r_a + r_c)}{r_b r_c} \le \frac{3R^3}{2r^3}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{(r_a + r_b)(r_a + r_c)}{r_b r_c} = \sum_{cyc} \frac{\left(\frac{rs}{s-a} + \frac{rs}{s-b}\right)\left(\frac{rs}{s-a} + \frac{rs}{s-c}\right)}{\frac{rs}{s-b} \cdot \frac{rs}{s-c}}$$

$$= \sum_{cyc} \frac{\frac{2s-a-b}{(s-a)(s-b)} \cdot \frac{2s-a-c}{(s-a)(s-c)}}{\frac{1}{(s-b)(s-c)}} = \sum_{cyc} \frac{bc}{(s-a)^2} \overset{A-G}{\le} \sum_{cyc} \frac{(b+c)^2}{4(s-a)^2} = \sum_{cyc} \frac{(s+s-a)^2}{4(s-a)^2}$$

$$= \sum_{cyc} \frac{s^2 + (s-a)^2 + 2s(s-a)}{4(s-a)^2} = \frac{3}{4} + \frac{1}{2r} \sum_{cyc} \frac{rs}{s-a} + \frac{1}{4r^2} \sum_{cyc} \frac{r^2 s^2}{(s-a)^2}$$

$$= \frac{3}{4} + \frac{1}{2r} \sum_{cyc} r_a + \frac{1}{4r^2} \sum_{cyc} r_a^2 = \frac{3}{4} + \frac{4R+r}{2r} + \frac{(4R+r)^2 - 2s^2}{4r^2}$$

$$= \frac{(4R+r)^2 - 2s^2 + 2r(4R+r) + 3r^2}{4r^2}$$

$$\overset{\text{Gerretsen}}{\le} \frac{(4R+r)^2 - 2(16Rr - 5r^2) + 2r(4R+r) + 3r^2}{4r^2} = \frac{4(R^2 - Rr + r^2)}{r^2} \overset{?}{\le} \frac{3R^3}{2r^3}$$

$$\Leftrightarrow 3t^3 - 8t^2 + 8t - 8 \ge 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)\left(2t^2 + t(t-2) + 4\right) \ge 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\ge} 2 \therefore \sum_{cyc} \frac{(r_a + r_b)(r_a + r_c)}{r_b r_c} \le \frac{3R^3}{2r^3}$$

$$\text{Again}, \sum_{cyc} \frac{(r_a + r_b)(r_a + r_c)}{r_b r_c} = \sum_{cyc} \frac{bc}{(s-a)^2} = \frac{1}{r} \sum_{cyc} \left(\frac{bc}{s(s-a)} \cdot \frac{rs}{s-a}\right)$$

$$= \frac{1}{r} \sum_{cyc} \left(\sec^2 \frac{A}{2} \cdot r_a\right) \overset{\text{Chebyshev}}{\ge} \frac{1}{3r} \left(\sum_{cyc} \sec^2 \frac{A}{2}\right)\left(\sum_{cyc} r_a\right)$$

$$\left(\because \text{WLOG assuming } a \ge b \ge c \Rightarrow \sec^2 \frac{A}{2} \ge \sec^2 \frac{B}{2} \ge \sec^2 \frac{C}{2} \text{ and } r_a \ge r_b \ge r_c\right)$$

$$= \frac{1}{3r} \cdot \frac{s^2 + (4R+r)^2}{s^2} \cdot (4R+r) \overset{?}{\ge} \frac{6R}{r} \Leftrightarrow (4R+r)^3 + s^2(4R+r) \overset{?}{\ge} 18Rs^2$$

$$\Leftrightarrow (4R+r)^3 \underset{(*)}{\overset{?}{\ge}} (14R-r)s^2, \text{but } (14R-r)s^2 \overset{\text{Gerretsen}}{\le}$$

$$(14R-r)(4R^2 + 4Rr + 3r^2) \overset{?}{\le} (4R+r)^3 \Leftrightarrow 4t^3 - 2t^2 - 13t + 2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (t-2)(4t^2 + 6t - 1) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \therefore \sum_{cyc} \frac{(r_a + r_b)(r_a + r_c)}{r_b r_c} \ge \frac{6R}{r}$$

$$\text{and hence}, \frac{6R}{r} \le \sum_{cyc} \frac{(r_a + r_b)(r_a + r_c)}{r_b r_c} \le \frac{3R^3}{2r^3}\ \forall\, \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\square\ \sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}=\frac{(r_a+r_b)(r_b+r_c)(r_a+r_c)}{r_ar_br_c}\cdot\sum_{cyc}\frac{r_a}{r_b+r_c}$$

$$=\frac{4Rs^2}{s^2r}\cdot\sum_{cyc}\frac{r_a}{r_b+r_c}\overset{Nesbitt}{\geq}\frac{6R}{r}.$$

$$\square\ \sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}=\frac{4R}{r}\cdot\sum_{cyc}\frac{r_a}{r_b+r_c}\overset{CBS}{\leq}\frac{4R}{r}\cdot\sum_{cyc}\frac{r_a}{4}\left(\frac{1}{r_b}+\frac{1}{r_c}\right)$$

$$=\frac{R}{r}\left(\sum_{cyc}r_a\cdot\sum_{cyc}\frac{1}{r_a}-3\right)$$

$$=\frac{R}{r}\left(\frac{4R+r}{r}-3\right)=\frac{3R^3}{2r^3}-\frac{R}{r}\left(\frac{R}{2r}-1\right)\left(\frac{3R}{r}-2\right)\overset{Euler}{\leq}\frac{3R^3}{2r^3}.$$

which completes the proof. Equality holds iff ΔABC is equilateral.