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In any Δ ABC, the following relationship holds :

$$3 \le \sum_{cyc} \frac{m_a h_a}{h_b h_c} \le 3\left(\frac{R}{2r}\right)^4$$

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$$\sum_{cyc} \frac{m_a h_a}{h_b h_c} = \sum_{cyc} \frac{m_a \frac{bc}{2R}}{\frac{ca.ab}{4R^2}} = 2R \sum_{cyc} \frac{m_a^2}{m_a a^2} \overset{\text{Tereshin}}{\le} 8R^2 \sum_{cyc} \frac{m_a^2}{a^2(b^2+c^2)}$$

$$= 2R^2 \sum_{cyc} \frac{2b^2+2c^2-a^2}{a^2(b^2+c^2)} = 4R^2.\frac{\sum_{cyc} b^2c^2}{a^2b^2c^2} - 2R^2 \sum_{cyc} \frac{1}{b^2+c^2} \overset{\substack{\text{Goldstone}\\ \text{and}\\ \text{Bergstrom}}}{\le}$$

$$\frac{4R^2.4R^2s^2}{16R^2r^2s^2} - 2R^2.\frac{9}{2\sum_{cyc} a^2} \overset{\text{Leibnitz}}{\le} \frac{R^2}{r^2} - \frac{2R^2.9}{2.9R^2} = \frac{R^2}{r^2} - 1 \overset{?}{\le} 3\left(\frac{R}{2r}\right)^4$$

$$\Leftrightarrow 3t^4 \overset{?}{\ge} 16t^2 - 16 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)\Big((t-2)(3t^2+12t+20)+32\Big) \overset{?}{\ge} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\ge} 2 \therefore \sum_{cyc} \frac{m_a h_a}{h_b h_c} \le 3\left(\frac{R}{2r}\right)^4$$

$$\text{Again}, \sum_{cyc} \frac{m_a h_a}{h_b h_c} = 2R \sum_{cyc} \frac{m_a}{a^2} \overset{\text{Tereshin}}{\ge} \frac{2R}{4R} \sum_{cyc} \frac{b^2+c^2}{a^2} = \frac{1}{2} \sum_{cyc} \left(\frac{b^2}{a^2} + \frac{a^2}{b^2}\right) \overset{A-G}{\ge} \frac{1}{2}.6$$

$$\therefore \sum_{cyc} \frac{m_a h_a}{h_b h_c} \ge 3 \therefore 3 \le \sum_{cyc} \frac{m_a h_a}{h_b h_c} \le 3\left(\frac{R}{2r}\right)^4$$

$\forall \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)