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In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{2h_bh_c}{9r^2}\ge 3+9r^2\sum_{cyc}\frac{1}{h_a^2}$$

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$$\sum_{cyc}\frac{2h_bh_c}{9r^2}\ge 3+9r^2\sum_{cyc}\frac{1}{h_a^2}\Leftrightarrow\sum_{cyc}\frac{2ca.ab}{4R^2.9r^2}\ge 3+\frac{9r^2}{4r^2s^2}\sum_{cyc}a^2$$

$$\Leftrightarrow\frac{2.4Rrs.2s}{4R^2.9r^2}\ge 3+\frac{9(s^2-4Rr-r^2)}{2s^2}\Leftrightarrow\frac{4s^2-27Rr}{9Rr}\ge\frac{9(s^2-4Rr-r^2)}{2s^2}$$

$$\Leftrightarrow 8s^4-135Rrs^2+81Rr^2(4R+r)\overset{(*)}{\ge}0\ and\ \because 8(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0$$

$\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\ge 8(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow(121R-80r)s^2\overset{(**)}{\ge}r(1724R^2-1361Rr+200r^2)$$

$$\text{Now, LHS of }(**)\overset{\text{Gerretsen}}{\ge}(121R-80r)(16Rr-5r^2)$$

$$\overset{?}{\ge}r(1724R^2-1361Rr+200r^2)\Leftrightarrow 53R^2-131Rr+50r^2\overset{?}{\ge}0$$

$$\Leftrightarrow(R-2r)(53R-25r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\frac{2h_bh_c}{9r^2}\ge 3+9r^2\sum_{cyc}\frac{1}{h_a^2}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$