In any Δ ABC, the $following$ $relationship$ holds :

$$F\left(4-\frac{2r}{R}\right)^2 \le \sum_{cyc} h_a^2 \cot\frac{A}{2} \le 9F\left(\frac{R}{2r}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} h_a^2 \cot\frac{A}{2} = \sum_{cyc}\left(\frac{b^2c^2}{4R^2}\cdot\frac{s-a}{r}\right)$$

$$= \frac{s\left(\left(s^2+4Rr+r^2\right)^2-16Rrs^2\right)-4Rrs\left(s^2+4Rr+r^2\right)}{4R^2r}$$

$$\Rightarrow \sum_{cyc} h_a^2 \cot\frac{A}{2} = \frac{s\left(s^4-\left(12Rr-2r^2\right)s^2+r^3(4R+r)\right)}{4R^2r} \to (1)$$

$$\therefore \text{ via } (1), \sum_{cyc} h_a^2 \cot\frac{A}{2} \le 9F\left(\frac{R}{2r}\right)^3$$

$$\Leftrightarrow \frac{s\left(s^4-\left(12Rr-2r^2\right)s^2+r^3(4R+r)\right)}{4R^2r} \le \frac{9rsR^3}{8r^3}$$

$$\Leftrightarrow 9R^5-8R^4r-2r^5+4r^2s^2(6R-r)-2rs^4 \overset{(*)}{\ge} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\ge} 9R^5-8R^4r-2r^5+4r^2s^2(6R-r)$$

$$-2rs^2\left(4R^2+4Rr+3r^2\right) \overset{?}{\ge} 0 \Leftrightarrow 2rs^2\left(4R^2-8Rr+5r^2\right) \underset{(**)}{\underbrace{\overset{?}{\le}}} 9R^5-8R^4r-2r^5$$

$$\text{Again, LHS of } (**) \overset{\text{Gerretsen}}{\le} 2r\left(4R^2-8Rr+5r^2\right)\left(4R^2+4Rr+3r^2\right) \overset{?}{\le}$$

$$9R^5-8R^4r-2r^5 \Leftrightarrow 9t^5-32t^4+32t^3-32 \overset{?}{\ge} 0 \ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(7t^3(t-2)+2t^4+4t^2+8t+16\right) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} h_a^2 \cot\frac{A}{2} \le 9F\left(\frac{R}{2r}\right)^3$$

$$\text{Also, via } (1), \sum_{cyc} h_a^2 \cot\frac{A}{2} \ge F\left(4-\frac{2r}{R}\right)^2$$

$$\Leftrightarrow \frac{s\left(s^4-\left(12Rr-2r^2\right)s^2+r^3(4R+r)\right)}{4R^2r} \ge \frac{rs(4R-2r)^2}{R^2}$$

$$\Leftrightarrow s^4-\left(12Rr-2r^2\right)s^2-r^2\left(64R^2-68Rr+15r^2\right) \underset{(***)}{\underbrace{\overset{?}{\ge}}} 0$$

$$\text{Now, } s^4-\left(12Rr-2r^2\right)s^2 \overset{\text{Gerretsen}}{\ge} \left(16Rr-5r^2\right)s^2-\left(12Rr-2r^2\right)s^2$$

$$= \left(4Rr-3r^2\right)s^2 \overset{\text{Gerretsen}}{\ge} \left(4Rr-3r^2\right)\left(16Rr-5r^2\right) = r^2\left(64R^2-68Rr+15r^2\right)$$

$$\Rightarrow (***) \text{ is true } \therefore \sum_{cyc} h_a^2 \cot\frac{A}{2} \ge F\left(4-\frac{2r}{R}\right)^2 \text{ and so,}$$

$$F\left(4-\frac{2r}{R}\right)^2 \le \sum_{cyc} h_a^2 \cot\frac{A}{2} \le 9F\left(\frac{R}{2r}\right)^3 \ \forall \Delta ABC,$$

$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$