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In any Δ ABC, the following relationship holds :

$$\sum_{cyc} h_a^2 \cot\frac{A}{2} \geq \frac{2r}{R}\sum_{cyc} r_a^2 \cot\frac{A}{2}$$

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$$\sum_{cyc} h_a^2 \cot\frac{A}{2} = \sum_{cyc}\left(\frac{b^2c^2}{4R^2}.\frac{s-a}{r}\right)$$

$$= \frac{s\left(\left(s^2+4Rr+r^2\right)^2-16Rrs^2\right)-4Rrs\left(s^2+4Rr+r^2\right)}{4R^2r}$$

$$\Rightarrow \sum_{cyc} h_a^2 \cot\frac{A}{2} = \frac{s\left(s^4-\left(12Rr-2r^2\right)s^2+r^3(4R+r)\right)}{4R^2r} \to (1)$$

$$\text{Also, } \sum_{cyc} r_a^2 \cot\frac{A}{2} = \sum_{cyc}\left(\frac{r^2s^2}{(s-a)^2}.\frac{s-a}{r}\right) = s\sum_{cyc} r_a$$

$$\Rightarrow \sum_{cyc} r_a^2 \cot\frac{A}{2} = s(4R+r) \to (2) \therefore (1),(2) \Rightarrow \sum_{cyc} h_a^2 \cot\frac{A}{2} \geq \frac{2r}{R}\sum_{cyc} r_a^2 \cot\frac{A}{2}$$

$$\Leftrightarrow \frac{s\left(s^4-\left(12Rr-2r^2\right)s^2+r^3(4R+r)\right)}{4R^2r} \geq \frac{2r}{R}.s(4R+r)$$

$$\Leftrightarrow s^4-\left(12Rr-2r^2\right)s^2-r^2\left(32R^2+4Rr-15r^2\right) \overset{(*)}{\geq} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} \left(16Rr-5r^2\right)s^2-\left(12Rr-2r^2\right)s^2$$

$$-r^2\left(32R^2+4Rr-15r^2\right) = \left(4Rr-3r^2\right)s^2-r^2\left(32R^2+4Rr-15r^2\right)$$

$$\overset{\text{Gerretsen}}{\geq} \left(4Rr-3r^2\right)\left(16Rr-5r^2\right)-r^2\left(32R^2+4Rr-15r^2\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow 4R^2-9Rr+2r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(4R-r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} h_a^2 \cot\frac{A}{2} \geq \frac{2r}{R}\sum_{cyc} r_a^2 \cot\frac{A}{2}\ \forall\,\Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$