ROMANIAN MATHEMATICAL MAGAZINE

In Δ ABC, m(BAC) = 90°, AD ⊥ BC. R_1, R_2 → circumradii and r_1, r_2 → inradii of Δ ABD, Δ ACD. Prove that :

$$\frac{r_a + r_b + r_c}{r + r_1 + r_2} \geq \left(\sqrt{2} + 1\right) \frac{r_a + r_b + r_c}{R + R_1 + R_2}$$

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$$2R\sin A = a, 2R_1 \sin \widehat{ADB} = c, 2R_2 \sin \widehat{ADC} = b$$

$$\therefore R = \frac{a}{2}, R_1 = \frac{c}{2}, R_2 = \frac{b}{2} \rightarrow (1)$$

$$\text{Now, } r_1 = 4R_1 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = c.\frac{1}{\sqrt{2}}.\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right)$$

$$= c.\frac{1}{\sqrt{2}}.\left(\frac{b+c}{a}.\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \frac{c}{2}.\frac{2(s-a)}{a} \overset{\text{via (1)}}{=} \frac{c}{a}.(2R + r - 2R)$$

$$\Rightarrow r_1 = \frac{c}{a}.r \text{ and analogously, } r_2 = \frac{b}{a}.r \rightarrow (2)$$

$$\text{Again, } \frac{R}{r} = \frac{R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{1}{\frac{2}{\sqrt{2}}.\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right)} \overset{0 < \cos\frac{B-C}{2} \leq 1}{\geq} \frac{1}{\sqrt{2}\left(1 - \frac{1}{\sqrt{2}}\right)}$$

$$= \frac{1}{\sqrt{2} - 1} \Rightarrow \frac{R}{r} \geq \sqrt{2} + 1 \rightarrow \text{(i)}$$

$$\text{Also, } r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{2R}{\sqrt{2}}.\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right) \overset{\text{via (1)}}{=} \frac{a}{\sqrt{2}}.\left(\frac{b+c}{a}.\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{a}{2}.\frac{2(s-a)}{a} \overset{\text{via (1)}}{=} 2R + r - 2R \Rightarrow r = s \rightarrow (3)$$

$$\therefore \frac{\left(\frac{r_a+r_b+r_c}{r+r_1+r_2}\right)}{\left(\frac{r_a+r_b+r_c}{R+R_1+R_2}\right)} \overset{\text{via (1),(2) and (3)}}{=} \frac{\frac{a}{2} + \frac{c}{2} + \frac{b}{2}}{s + \frac{c}{a}.s + \frac{b}{a}.s} = \frac{s}{s\left(1 + \frac{b}{a} + \frac{c}{a}\right)} = \frac{a}{a+b+c}$$

$$\overset{\text{via (1)}}{=} \frac{2R}{2s} \overset{\text{via (3)}}{=} \frac{R}{r} \overset{\text{via (1)}}{\geq} \sqrt{2} + 1, \text{"="} \text{ iff } \Delta \text{ ABC is isosceles right} - \text{angled (QED)}$$