ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{r_a}{r_a+\lambda r_b}\le\frac{3}{\lambda+1}\cdot\frac{4(\lambda+1)R+(1-8\lambda)r}{4R+r}\text{ for }\frac{1}{2}\le\lambda\le\frac{7}{2}$$

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$$\sum_{cyc}\frac{r_a}{r_a+\lambda r_b}\le\frac{3}{\lambda+1}\cdot\frac{4(\lambda+1)R+(1-8\lambda)r}{4R+r}$$

$$\Leftrightarrow\sum_{cyc}\frac{r_a+\lambda r_b-\lambda r_b}{r_a+\lambda r_b}\le\frac{3}{\lambda+1}\cdot\frac{4R+r+\lambda(4R-8r)}{4R+r}$$

$$\Leftrightarrow 3-\frac{3}{\lambda+1}-\lambda\sum_{cyc}\frac{r_b^2}{r_ar_b+\lambda r_b^2}\le\frac{3\lambda(4R-8r)}{(\lambda+1)(4R+r)}$$

$$\Leftrightarrow\frac{3\lambda}{\lambda+1}-\lambda\sum_{cyc}\frac{r_b^2}{r_ar_b+\lambda r_b^2}-\frac{3\lambda(4R-8r)}{(\lambda+1)(4R+r)}\le 0$$

$$\Leftrightarrow\sum_{cyc}\frac{r_b^2}{r_ar_b+\lambda r_b^2}-\frac{3}{\lambda+1}+\frac{3(4R-8r)}{(\lambda+1)(4R+r)}\boxed{\overset{(*)}{\ge}}0\ (\because\lambda>0)$$

$$\text{Now, }\sum_{cyc}\frac{r_b^2}{r_ar_b+\lambda r_b^2}-\frac{3}{\lambda+1}\overset{\text{Bergstrom}}{\ge}\frac{(4R+r)^2}{s^2+\lambda((4R+r)^2-2s^2)}-\frac{3}{\lambda+1}$$

$$=\frac{\lambda(4R+r)^2+(4R+r)^2-3s^2-3\lambda\left((4R+r)^2-2s^2\right)}{(\lambda+1)\left(s^2+\lambda((4R+r)^2-2s^2)\right)}$$

$$=\frac{(4R+r)^2-3s^2-2\lambda\left((4R+r)^2-3s^2\right)}{(\lambda+1)\left(s^2+\lambda((4R+r)^2-2s^2)\right)}$$

$$=\frac{(1-2\lambda)\left((4R+r)^2-3s^2\right)}{(\lambda+1)\left(s^2+\lambda((4R+r)^2-2s^2)\right)}\therefore\text{ in order to prove }(*)\text{, it suffices to prove :}$$

$$\frac{(2\lambda-1)\left((4R+r)^2-3s^2\right)}{(\lambda+1)\left(s^2+\lambda((4R+r)^2-2s^2)\right)}\le\frac{3(4R-8r)}{(\lambda+1)(4R+r)}$$

$$\Leftrightarrow 3(4R-8r)\left(\lambda(4R+r)^2-(2\lambda-1)s^2\right)\ge(2\lambda-1)(4R+r)\left((4R+r)^2-3s^2\right)$$

$$\Leftrightarrow 3\lambda(4R-8r)(4R+r)^2\ge(2\lambda-1)\left((4R+r)^3-3s^2(4R+r)+3(4R-8r)s^2\right)$$

$$\Leftrightarrow 3\lambda(4R-8r)(4R+r)^2\ge(2\lambda-1)\left((4R+r)^3-27rs^2\right)$$

$$\Leftrightarrow(4R+r)^3-27rs^2+\lambda\left(3(4R-8r)(4R+r)^2-2(4R+r)^3+54rs^2\right)\boxed{\boxed{\overset{(**)}{\ge}}}0$$

$\boxed{\text{Case 1}}$ $3(4R-8r)(4R+r)^2-2(4R+r)^3+54rs^2\ge 0$ *and then* : LHS of $(**)\ge$

$$(4R+r)^3-27rs^2\overset{\text{Trucht}}{\ge}3s^2(4R+r-9r)=3s^2(4R-8r)\overset{\text{Euler}}{\ge}0\Rightarrow(**)\text{ is true}$$

$\boxed{\text{Case 2}}$ $3(4R-8r)(4R+r)^2-2(4R+r)^3+54rs^2<0$ *and then* : LHS of $(**)=$

$$(4R+r)^3-27rs^2-\lambda\left(-\left(3(4R-8r)(4R+r)^2-2(4R+r)^3+54rs^2\right)\right)\overset{0<\lambda\le\frac{7}{2}}{\ge}$$

$$(4R+r)^3-27rs^2-\frac{7}{2}\left(-\left(3(4R-8r)(4R+r)^2-2(4R+r)^3+54rs^2\right)\right)\overset{?}{\geq}0$$

$$\Leftrightarrow 2(4R+r)^3-54rs^2+21(4R-8r)(4R+r)^2-14(4R+r)^3+7.54rs^2\overset{?}{\geq}0$$

$$\Leftrightarrow (4R+r)^2(84R-168r-48R-12r)+6.54rs^2\overset{?}{\geq}0$$

$$\Leftrightarrow (R-5r)(4R+r)^2+9rs^2\boxed{\underset{(***)}{\overset{?}{\geq}}}0$$

$$\text{Now, LHS of }(***)\overset{\text{Gerretsen}}{\geq}(R-5r)(4R+r)^2+9r\left(16Rr-5r^2\right)\overset{?}{\geq}0\Leftrightarrow$$

$$16t^3-72t^2+105t-50\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(4t-5)^2\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$\Rightarrow (***) \Rightarrow (**)$ **is true** $\therefore$ **combining both cases,** $(**) \Rightarrow (*)$ **is true** $\forall\,\Delta\,ABC$ **with** $\frac{1}{2}\leq\lambda\leq\frac{7}{2}\therefore\sum_{cyc}\frac{r_a}{r_a+\lambda r_b}\leq\frac{3}{\lambda+1}.\frac{4(\lambda+1)R+(1-8\lambda)r}{4R+r}$ $\forall\,\Delta\,ABC$ **with** $\frac{1}{2}\leq\lambda\leq\frac{7}{2}$,

$''='' $ **iff** $\Delta\,ABC$ **is equilateral and** $\lambda=\frac{7}{2}$ **(QED)**