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In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{h_b+h_c}\ge 9R$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}\text{ and analogs}$$

$$\therefore \text{via (i)}, \sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{h_b+h_c}=\sum_{cyc}\frac{4R\cos^2\frac{C}{2}.4R\cos^2\frac{B}{2}}{\frac{ca+ab}{2R}}$$

$$=\sum_{cyc}\frac{4R\cos^2\frac{C}{2}.4R\cos^2\frac{B}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}.4R\cos\frac{A}{2}\cos\frac{B-C}{2}}\overset{0<\cos\frac{B-C}{2}\le 1}{\ge}2R\sum_{cyc}\frac{\cos^2\frac{B}{2}\cos^2\frac{C}{2}}{\cos^2\frac{A}{2}\sin\frac{A}{2}}\overset{A-G}{\ge}6R.\sqrt[3]{\frac{\prod_{cyc}\cos^2\frac{A}{2}}{\prod_{cyc}\sin\frac{A}{2}}}$$

$$=6R.\sqrt[3]{\frac{\frac{s^2}{16R^2}}{\frac{r}{4R}}}=6R.\sqrt[3]{\frac{2s^2}{8Rr}}\overset{\text{Gerretsen}}{\ge}6R.\sqrt[3]{\frac{27Rr+5r(R-2r)}{8Rr}}\overset{\text{Euler}}{\ge}6R.\sqrt[3]{\frac{27Rr}{8Rr}}=9R$$

$$\therefore \sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{h_b+h_c}\ge 9R\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$