ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{r_a^2}{r_a+r_b}+\frac{r_b^2}{r_b+r_c}\ge\frac{1}{8r}\left(a^2-3c^2-ab+2ac+3bc\right)$$

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$$\frac{r_a^2}{r_a+r_b}+\frac{r_b^2}{r_b+r_c}=\frac{\frac{F^2}{(s-a)^2}}{\frac{F}{s-a}+\frac{F}{s-b}}+\frac{\frac{F^2}{(s-b)^2}}{\frac{F}{s-b}+\frac{F}{s-c}}=\frac{F(s-b)}{c(s-a)}+\frac{F(s-c)}{a(s-b)}$$

$$\Rightarrow 8r\left(\frac{r_a^2}{r_a+r_b}+\frac{r_b^2}{r_b+r_c}\right)=8\left(\frac{F^2(s-b)}{sc(s-a)}+\frac{F^2(s-c)}{sa(s-b)}\right)$$

$$=8\left(\frac{s(s-a)(s-b)(s-c)(s-b)}{sc(s-a)}+\frac{s(s-a)(s-b)(s-c)(s-c)}{sa(s-b)}\right)$$

$$=8\left(\frac{(s-b)^2(s-c)}{c}+\frac{(s-c)^2(s-a)}{a}\right)=\frac{8y^2z}{x+y}+\frac{8z^2x}{y+z}$$

$$(x=s-a,y=s-b,z=s-c\Rightarrow a=y+z,b=z+x,c=x+y)$$

$$=\frac{8y^2z(y+z)+8z^2x(x+y)}{(x+y)(y+z)}\ge a^2-3c^2-ab+2ac+3bc$$

$$\Leftrightarrow\frac{8y^2z(y+z)+8z^2x(x+y)}{(x+y)(y+z)}\ge$$

$$(y+z)^2-3(x+y)^2-(y+z)(z+x)+2(y+z)(x+y)+3(z+x)(x+y)$$

$$\Leftrightarrow x^2y^2+2z^2x^2+xy^3+y^3z+y^2z^2\overset{(*)}{\ge}x^2yz+4xy^2z+xyz^2$$

Now, $x^2y^2+y^2z^2+z^2x^2\ge x^2yz+xy^2z+xyz^2\to(1)$ *and*

$z^2x^2+xy^3+y^3z\overset{A-G}{\ge}3xy^2z\to(2)$ *and* $(1)+(2)\Rightarrow(*)$ is true

$$\therefore\frac{r_a^2}{r_a+r_b}+\frac{r_b^2}{r_b+r_c}\ge\frac{1}{8r}\left(a^2-3c^2-ab+2ac+3bc\right)$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)