ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$3.\sqrt[3]{\frac{r^2}{2R^2}} \le \sum_{cyc} \frac{h_a}{\sqrt{(r_a+r_b)(r_a+r_c)}} \le \frac{4R+r}{6r}$$

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$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2} \text{ and analogs}$$

$$\text{Now, } \sum_{cyc} \frac{h_a}{\sqrt{(r_a+r_b)(r_a+r_c)}} \overset{\text{via (i)}}{=} \sum_{cyc} \frac{\frac{2rs}{a}.\cos\frac{A}{2}}{4R\cos\frac{C}{2}\cos\frac{B}{2}\cos\frac{A}{2}} = \sum_{cyc} \frac{\frac{2rs\cos\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}}{4R.\frac{s}{4R}}$$

$$= \frac{r}{2R}\sum_{cyc}\frac{1}{\sin\frac{A}{2}} = \frac{r}{2R}\sum_{cyc}\sqrt{\frac{bc(s-a)}{r^2s}} \overset{CBS}{\le} \frac{1}{2R.\sqrt{s}}.\sqrt{\sum_{cyc}(s-a)}.\sqrt{s^2+4Rr+r^2}$$

$$\overset{\text{Gerretsen}}{\le} \frac{1}{2R}.\sqrt{4R^2+8Rr+4r^2} = \frac{R+r}{R} \overset{?}{\le} \frac{4R+r}{6r} \Leftrightarrow 4R^2-5Rr-6r^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (4R+3r)(R-2r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \therefore \sum_{cyc}\frac{h_a}{\sqrt{(r_a+r_b)(r_a+r_c)}} \le \frac{4R+r}{6r}$$

$$\text{Again, } \sum_{cyc}\frac{h_a}{\sqrt{(r_a+r_b)(r_a+r_c)}} = \frac{r}{2R}\sum_{cyc}\frac{1}{\sin\frac{A}{2}} \overset{A-G}{\ge} \frac{3r}{2R}.\sqrt[3]{\frac{4R}{r}} = \frac{3}{2R}.\sqrt[3]{\frac{r^3.4R.2R^2}{r.2R^2}}$$

$$= \frac{3.2R}{2R}.\sqrt[3]{\frac{r^2}{2R^2}} \therefore \sum_{cyc}\frac{h_a}{\sqrt{(r_a+r_b)(r_a+r_c)}} \ge 3.\sqrt[3]{\frac{r^2}{2R^2}} \text{ and so,}$$

$$3.\sqrt[3]{\frac{r^2}{2R^2}} \le \sum_{cyc}\frac{h_a}{\sqrt{(r_a+r_b)(r_a+r_c)}} \le \frac{4R+r}{6r}$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)