ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{(b^2+c^2)h_a}{b+c}\le\sum_{cyc}\frac{(b^2+c^2)r_a}{b+c}$$

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$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\sum_{cyc}\frac{1}{a(b+c)}=\sum_{cyc}\frac{bc\left(\sum_{cyc}ab+a^2\right)}{4Rrs.2s(s^2+2Rr+r^2)}=\frac{\left(\sum_{cyc}ab\right)^2+8Rrs^2}{8Rrs^2(s^2+2Rr+r^2)}$$

$$\Rightarrow\sum_{cyc}\frac{1}{a(b+c)}\overset{(\bullet)}{=}\frac{\left(s^2+4Rr+r^2\right)^2+8Rrs^2}{8Rrs^2(s^2+2Rr+r^2)}$$

$$\sum_{cyc}\frac{a}{b+c}=\sum_{cyc}\frac{a-2s+2s}{2s-a}=-3+\frac{2s}{2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}\left(\sum_{cyc}ab+a^2\right)$$

$$=-3+\frac{2s}{2s(s^2+2Rr+r^2)}\cdot\left(\left(\sum_{cyc}a\right)^2+\sum_{cyc}ab\right)=-3+\frac{2s\left(5s^2+4Rr+r^2\right)}{2s(s^2+2Rr+r^2)}$$

$$\Rightarrow\sum_{cyc}\frac{a}{b+c}\overset{(\bullet\bullet)}{=}\frac{2s^2-2Rr-2r^2}{s^2+2Rr+r^2}$$

$$\sum_{cyc}\mathrm{cosec}^2\frac{A}{2}=3+\sum_{cyc}\cot^2\frac{A}{2}=3+\sum_{cyc}\frac{s^2}{r_a^2}$$

$$=3+\frac{s^2}{r^2s^4}\left(\left(\sum_{cyc}r_br_c\right)^2-2r_ar_br_c\left(\sum_{cyc}r_a\right)\right)=3+\frac{s^2\left(s^4-2rs^2(4R+r)\right)}{r^2s^4}$$

$$\Rightarrow\sum_{cyc}\mathrm{cosec}^2\frac{A}{2}\overset{(\bullet\bullet\bullet)}{=}\frac{s^2-8Rr+r^2}{r^2}$$

$$\sum_{cyc}b^2c^2r_a\overset{\text{via (i)}}{=}\sum_{cyc}b^2c^2\left(4R+r-4R\cos^2\frac{A}{2}\right)$$

$$=(4R+r)\sum_{cyc}b^2c^2-\sum_{cyc}\frac{16R^2r^2s^2.4R\cos^2\frac{A}{2}}{16R^2\cos^2\frac{A}{2}\sin^2\frac{A}{2}}$$

$$= (4R+r)\sum_{cyc} b^2c^2 - 4Rr^2s^2\sum_{cyc}\mathrm{cosec}^2\frac{A}{2}$$

$$\overset{\text{via }(\bullet\bullet\bullet)}{=} (4R+r)\sum_{cyc} a^2b^2 - 4Rr^2s^2.\frac{s^2-8Rr+r^2}{r^2}$$

$$\Rightarrow \sum_{cyc} b^2c^2r_a \overset{(\bullet\bullet\bullet\bullet)}{=} (4R+r)\sum_{cyc} a^2b^2 - 4Rs^2(s^2-8Rr+r^2)$$

$$\sum_{cyc} a^2r_a = rs\sum_{cyc}\frac{(a-s+s)^2}{s-a} = rs\left(\sum_{cyc}(s-a) - 6s + \frac{s^2}{r^2s}\sum_{cyc}(s-b)(s-c)\right)$$

$$= rs\left(-5s + \frac{s^2(4Rr+r^2)}{r^2s}\right) \Rightarrow \sum_{cyc} a^2r_a \overset{(\bullet\bullet\bullet\bullet\bullet)}{=} (4R-4r)s^2$$

$$\therefore \sum_{cyc}\frac{(b^2+c^2)h_a}{b+c} = \sum_{cyc}\frac{\left(\sum_{cyc}a^2 - a^2\right)h_a}{b+c}$$

$$= 2rs\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{1}{a(b+c)}\right) - 2rs\sum_{cyc}\frac{a}{b+c}$$

$$\overset{\text{via }(\bullet),(\bullet\bullet)}{=} \frac{2rs\left((s^2-4Rr-r^2)\left((s^2+4Rr+r^2)^2+8Rrs^2\right) - 4Rrs^2(2s^2-2Rr-2r^2)\right)}{4Rrs^2(s^2+2Rr+r^2)}$$

$$\Rightarrow \boxed{\sum_{cyc}\frac{(b^2+c^2)h_a}{b+c} \overset{(1)}{=} \frac{s^6+(4Rr+r^2)s^4 - r^2s^2(40R^2+8Rr+r^2) - (4Rr+r^2)^3}{2Rs(s^2+2Rr+r^2)}} \text{ and}$$

$$\sum_{cyc}\frac{(b^2+c^2)r_a}{b+c} = \sum_{cyc}\frac{\left(a^2+\sum_{cyc}ab\right)(b^2+c^2)r_a}{2s(s^2+2Rr+r^2)}$$

$$= \frac{1}{2s(s^2+2Rr+r^2)}\left(\begin{array}{c}\sum_{cyc}\left(\left(\sum_{cyc}a^2b^2 - b^2c^2\right)r_a\right) + \\ \left(\sum_{cyc}ab\right)\sum_{cyc}\left(\left(\sum_{cyc}a^2 - a^2\right)r_a\right)\end{array}\right)$$

$$= \frac{1}{2s(s^2+2Rr+r^2)}.\left(\begin{array}{c}\left(\sum_{cyc}a^2b^2\right)(4R+r) - \sum_{cyc}b^2c^2r_a + \\ \left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)(4R+r) - \left(\sum_{cyc}ab\right)\sum_{cyc}a^2r_a\end{array}\right)$$

$$\overset{via\ (\bullet\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet\bullet)}{=} \frac{1}{2s(s^2+2Rr+r^2)}\left(\begin{array}{c}\left(\sum_{cyc} a^2b^2\right)(4R+r)-(4R+r)\sum_{cyc} a^2b^2+\\ 4Rs^2(s^2-8Rr+r^2)+\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)(4R+r)-\\ \left(\sum_{cyc} ab\right)(4R-4r)s^2\end{array}\right)$$

$$\Rightarrow \boxed{\sum_{cyc}\frac{(b^2+c^2)r_a}{b+c} \overset{(2)}{=} \frac{(4R+3r)s^4-rs^2(24R^2-8Rr-2r^2)-r^2(4R+r)^3}{s(s^2+2Rr+r^2)}}$$

$$\therefore (1),(2) \Rightarrow \sum_{cyc}\frac{(b^2+c^2)h_a}{b+c} \le \sum_{cyc}\frac{(b^2+c^2)r_a}{b+c}$$

$$\Leftrightarrow \frac{s^6+(4Rr+r^2)s^4-r^2s^2(40R^2+8Rr+r^2)-(4Rr+r^2)^3}{2Rs(s^2+2Rr+r^2)} \le \frac{(4R+3r)s^4-rs^2(24R^2-8Rr-2r^2)-r^2(4R+r)^3}{s(s^2+2Rr+r^2)}$$

$$\Leftrightarrow \boxed{\begin{array}{c}s^6-(8R^2+2Rr-r^2)s^4+rs^2(48R^3-56R^2r-12Rr^2-r^3)+\\ r^2(128R^4+32R^3r-24R^2r^2-10Rr^3-r^4)\overset{(*)}{\le} 0\end{array}}$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\le} (4R^2+4Rr+3r^2)s^4-(8R^2+2Rr-r^2)s^4 +rs^2(48R^3-56R^2r-12Rr^2-r^3)$$
$$+r^2(128R^4+32R^3r-24R^2r^2-10Rr^3-r^4)\overset{?}{\le} 0$$
$$\Leftrightarrow (4R^2-2Rr-4r^2)s^4-rs^2(48R^3-56R^2r-12Rr^2-r^3) \underbrace{\overset{?}{\ge}}_{(**)} r^2(128R^4+32R^3r-24R^2r^2-10Rr^3-r^4)$$

$$\text{Again, LHS of } (**) \overset{\text{Gerretsen}}{\ge} (4R^2-2Rr-4r^2)(16Rr-5r^2)s^2 -rs^2(48R^3-56R^2r-12Rr^2-r^3)=r(16R^3+4R^2r-42Rr^2+21r^3)s^2$$
$$\overset{?}{\ge} r^2(128R^4+32R^3r-24R^2r^2-10Rr^3-r^4) \Leftrightarrow$$
$$(16R^3+4R^2r-42Rr^2+21r^3)s^2 \underbrace{\overset{?}{\ge}}_{(***)} r(128R^4+32R^3r-24R^2r^2-10Rr^3-r^4)$$

$$\text{Once again, LHS of } (***) \overset{\text{Gerretsen}}{\ge}$$
$$(16R^3+4R^2r-42Rr^2+21r^3)(16Rr-5r^2)\overset{?}{\ge} r(128R^4+32R^3r-24R^2r^2-10Rr^3-r^4)$$
$$\Leftrightarrow 32t^4-12t^3-167t^2+139t-26\overset{?}{\ge} 0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(32t^3+20t^2+32t(t-2)+t+13)\overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$
$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{(b^2+c^2)h_a}{b+c} \le \sum_{cyc}\frac{(b^2+c^2)r_a}{b+c}\ \forall\ \Delta\,ABC,'' ='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$