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In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{r_br_c}{rp^2+r_a^3}\ge\frac{1}{2r}$$

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$$\sum_{cyc}\frac{r_br_c}{rp^2+r_a^3}=\sum_{cyc}\frac{r_br_c+r_a^2-r_a^2}{r_ar_br_c+r_a^3}=\sum_{cyc}\frac{r_br_c+r_a^2-r_a^2}{r_a\left(r_br_c+r_a^2\right)}$$

$$=\sum_{cyc}\frac{1}{r_a}-\sum_{cyc}\frac{r_a}{r_br_c+r_a^2}\overset{A-G}{\ge}\frac{1}{r}-\sum_{cyc}\frac{r_a}{2r_a.\sqrt{r_br_c}}\ge\frac{1}{r}-\frac{1}{2}\sum_{cyc}\frac{1}{w_a}\ge\frac{1}{r}-\frac{1}{2}\sum_{cyc}\frac{1}{h_a}$$

$$=\frac{1}{r}-\frac{1}{2r}=\frac{1}{2r}\therefore\sum_{cyc}\frac{r_br_c}{rp^2+r_a^3}\ge\frac{1}{2r}\ \forall\,\Delta\,ABC,''=''\ \text{iff }\Delta\text{ ABC is equilateral (QED)}$$