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In ΔABC holds:

$$\sum \cos^3\frac{B-C}{2} \geq \frac{16r}{\sqrt{3}\,R}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

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Solution by Tapas Das-India

$$\prod \cos\frac{B-C}{2} = \frac{(a+b)(b+c)(c+a)}{abc}\prod \sin\frac{A}{2} = \frac{s^2+2Rr+r^2}{8R^2}$$

$$\sum \cos^3\frac{B-C}{2} \overset{AM-GM}{\geq} 3\prod \cos\frac{B-C}{2}$$

$$\frac{16r}{\sqrt{3}\,R}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{16r}{\sqrt{3}\,R}\frac{s}{4R} \overset{Mitrinovic}{\leq} \frac{48Rr}{8R^2}$$

We need to show:

$$\frac{3(s^2+2Rr+r^2)}{8R^2} \geq \frac{48Rr}{8R^2}, \qquad s^2 \geq 14Rr - r^2$$

$$s^2 \geq 16Rr - 5r^2 \geq 14Rr - r^2 (Gerretsen) or\ R \geq 2r\ (Euler)$$