ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$(a^n+b^n+c^n)(a^n\cos^2 A+b^n\cos^2 B+c^n\cos^2 C)\ge 4F^2.\sum_{cyc} b^{n-2}c^{n-2},\ \ n\in\mathbb{N}$$

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Let H be the orthocenter of ΔABC. We have $\left(a^n.\overrightarrow{HA}+b^n.\overrightarrow{HB}+c^n.\overrightarrow{HC}\right)^2\ge 0$, then

$$a^{2n}.HA^2+b^{2n}.HB^2+c^{2n}.HC^2+a^nb^n.2\overrightarrow{HA}.\overrightarrow{HB}+b^nc^n.2\overrightarrow{HB}.\overrightarrow{HC}+c^na^n.2\overrightarrow{HC}.\overrightarrow{HA}\ge 0$$

$$\Leftrightarrow a^{2n}.HA^2+b^{2n}.HB^2+c^{2n}.HC^2+a^nb^n.\left(HA^2+HB^2-c^2\right)+b^nc^n.\left(HB^2+HC^2-a^2\right)+$$

$$+c^na^n.\left(HC^2+HA^2-b^2\right)\ge 0$$

$$\Leftrightarrow (a^n+b^n+c^n)\left(a^nHA^2+b^nHB^2+c^nHC^2\right)\ge a^2b^2c^2\left(a^{n-2}b^{n-2}+b^{n-2}c^{n-2}+c^{n-2}a^{n-2}\right)$$

Since $HA=2R|\cos A|$ (and analogs) and $abc=4RF$, then

$$(a^n+b^n+c^n)\left(a^n\cos^2 A+b^n\cos^2 B+c^n\cos^2 C\right)\ge 4F^2\left(a^{n-2}b^{n-2}+b^{n-2}c^{n-2}+c^{n-2}a^{n-2}\right),$$

as desired. Equality holds iff ΔABC is equilateral.