ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC, the following relationship holds :

$$9\sum_{cyc}\tan^3\frac{A}{2}\tan\frac{B}{2}+\frac{(4R+r)^2}{p^2}\geq 6$$

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Solution by Soumava Chakraborty-Kolkata-India

$$9\sum_{cyc}\tan^3\frac{A}{2}\tan\frac{B}{2}=9\sum_{cyc}\frac{\tan^3\frac{A}{2}\tan^3\frac{B}{2}}{\tan^2\frac{B}{2}}\overset{Radon}{\geq}9.\frac{\left(\sum_{cyc}\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)\right)^3}{\left(\sum_{cyc}\tan\frac{A}{2}\right)^2}$$

$$=9.\frac{\left(\frac{1}{p^2}\sum_{cyc}r_ar_b\right)^3}{\left(\frac{1}{p}\sum_{cyc}r_a\right)^2}=9.\frac{\left(\frac{1}{p^2}.p^2\right)^3}{\left(\frac{4R+r}{p}\right)^2}=\frac{9p^2}{(4R+r)^2}\Rightarrow$$

$$9\sum_{cyc}\tan^3\frac{A}{2}\tan\frac{B}{2}+\frac{(4R+r)^2}{p^2}\geq\frac{9p^2}{(4R+r)^2}+\frac{(4R+r)^2}{p^2}\overset{A-G}{\geq}$$

$$\geq 2.\sqrt{\frac{9p^2}{(4R+r)^2}.\frac{(4R+r)^2}{p^2}}=6$$

$$\therefore 9\sum_{cyc}\tan^3\frac{A}{2}\tan\frac{B}{2}+\frac{(4R+r)^2}{p^2}\geq 6\ \forall\ \Delta ABC,''='' \text{ iff } \Delta ABC \text{ is equilateral } (QED)$$