ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC , $m(BAC)=90^\circ, AD\perp BC$, R_1,R_2-circumradii and r_1,r_2-inradii

Prove that:

$$\frac{r_a+r_b+r_c}{r+r_1+r_2}\geq\left(\sqrt{2}+1\right)\frac{r_a+r_b+r_c}{R+R_1+R_2}$$

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Obviously,we have to prove that:

$$\frac{R+R_1+R_2}{r+r_1+r_2}\geq\sqrt{2}+1 \quad (1)$$

It is known that:

In $\Delta ABD\ (\measuredangle D=90^\circ)\quad R_1=\frac{c}{2}, r_1=\frac{m+h-c}{2}$

In $\Delta ACD\ (\measuredangle D=90^\circ)\quad R_2=\frac{b}{2}, r_2=\frac{m+h-b}{2}$

In $\Delta BAC\ (\measuredangle A=90^\circ)\quad R=\frac{a}{2}, r=\frac{b+c-a}{2}$

Consider the given (1)*:*

$$\frac{R+R_1+R_2}{r+r_1+r_2}=\frac{\frac{a}{2}+\frac{c}{2}+\frac{b}{2}}{\frac{b+c-a}{2}+\frac{m+h-c}{2}+\frac{n+h-b}{2}}=$$

$$=\frac{a+b+c}{b+c-a+m+h-c+n+h-b}=\frac{a+b+c}{\underbrace{(m+n)}_{a}+2h-a}=$$

$$=\frac{a+b+c}{2h}=\frac{2p}{2h}=\frac{p}{h}=\frac{\frac{3}{r}}{\frac{2S}{a}}=\frac{a}{2r}$$

$$\frac{R+R_1+R_2}{r+r_1+r_2}=\frac{a}{2r}=\frac{a}{b+c-a}=\frac{2R}{2RsinB+2RsinC-2R}=\frac{2R}{2R(sinB+sinC-1)}=$$

$$=\frac{1}{sinB+sinC-1}=\frac{1}{2sin\frac{B+C}{2}\cdot cos\frac{B-C}{2}-1}=\frac{1}{\sqrt{2}cos\frac{B-C}{2}-1}$$

$$sin\frac{B+C}{2}=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}\ ;\ 0\leq\cos\frac{B-C}{2}\leq 1$$

Therefore

$$\frac{R+R_1+R_2}{r+r_1+r_2}=\frac{1}{\sqrt{2}cos\frac{B-C}{2}-1}\geq\frac{1}{\sqrt{2}-1}=\sqrt{2}+1\ \ (qed)$$

So,

$$\frac{R+R_1+R_2}{r+r_1+r_2}\geq\sqrt{2}+1$$