ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{1}{1+\cot\frac{B}{2}.\cot\frac{C}{2}}\le\frac{3}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{1+\cot\frac{B}{2}.\cot\frac{C}{2}}=\sum_{cyc}\frac{1}{1+\frac{s^2}{r_br_c}}=\sum_{cyc}\frac{s(s-a)}{s(s-a)+s^2}=\sum_{cyc}\frac{2s-a-s}{2s-a}$$

$$=3-\sum_{cyc}\frac{s}{b+c}=3-s.\frac{\sum_{cyc}\left(\sum_{cyc}ab+a^2\right)}{2s(s^2+2Rr+r^2)}=3-\frac{\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab}{2(s^2+2Rr+r^2)}$$

$$=3-\frac{4s^2+s^2+4Rr+r^2}{2(s^2+2Rr+r^2)}=\frac{s^2+8Rr+5r^2}{2(s^2+2Rr+r^2)}\overset{?}{\le}\frac{3}{4}\Leftrightarrow s^2-10Rr-7r^2\overset{?}{\ge}0$$

$$\Leftrightarrow s^2-16Rr+5r^2+6r(R-2r)\overset{?}{\ge}0\to\text{true}\because s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\ge}0\text{ and}$$

$$6r(R-2r)\overset{\text{Euler}}{\ge}0\therefore\sum_{cyc}\frac{1}{1+\cot\frac{B}{2}.\cot\frac{C}{2}}\le\frac{3}{4}$$

$\forall\,\Delta\,ABC,''=''$ iff Δ ABC is equilateral (QED)