ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{9}{8} \le \sum_{cyc}\left(\frac{r_a}{b+c}.\sin A\right) \le \frac{9R}{16r}$$

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$$\sum_{cyc}\left(\frac{r_a}{b+c}.\sin A\right) = \frac{1}{2R}.\sum_{cyc}\frac{rsa}{(s-a)(2s-a)}$$

$$= \frac{r}{2R}.\sum_{cyc}\frac{a\big((2s-a)-(s-a)\big)}{(s-a)(2s-a)} = \frac{r}{2R}\left(\sum_{cyc}\frac{a-s+s}{s-a} - \sum_{cyc}\frac{a}{b+c}\right)$$

$$= \frac{r}{2R}\left(-3+\frac{s(4Rr+r^2)}{r^2s} - \sum_{cyc}\frac{a}{b+c}\right) \overset{(1)}{=} \frac{r}{2R}\left(\frac{4R-2r}{r} - \sum_{cyc}\frac{a}{b+c}\right)$$

$$= \frac{r}{2R}\left(\frac{4R-2r}{r} - \sum_{cyc}\frac{2s-(b+c)}{b+c}\right) = \frac{r}{2R}\left(\frac{4R-2r}{r} + 3 - 2s.\frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}\right)$$

$$= \frac{(4R-4r)s^2+2Rr(4R+r)}{2R(s^2+2Rr+r^2)} \overset{?}{\ge} \frac{9}{8} \Leftrightarrow \underbrace{(7R-16r)s^2+Rr(14R-r) \overset{?}{\ge} 0}_{(*)}$$

Case 1 $7R-16r \ge 0$ *and then* : LHS of $(*) \ge Rr(14R-r) > 0 \Rightarrow (*)$ is true (strict inequality)

Case 2 $7R-16r < 0$ *and then* : LHS of $(*) \ge (7R-16r)(4R^2+4Rr+3r^2)$

$$+Rr(14R-r) \overset{?}{\ge} 0 \Leftrightarrow 14t^3-11t^2-22t-24 \overset{?}{\ge} 0 \ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(14t^2+17t+12) \overset{?}{\ge} 0 \to \text{true} \because t \overset{Euler}{\ge} 2 \Rightarrow (*) \text{ is true}$$

$\therefore$ combining both cases, $(*)$ is true for *all* triangles $\therefore \sum_{cyc}\left(\frac{r_a}{b+c}.\sin A\right) \ge \frac{9}{8}$

$$\text{Again, } \sum_{cyc}\left(\frac{r_a}{b+c}.\sin A\right) \overset{via\,(1)}{=} \frac{r}{2R}\left(\frac{4R-2r}{r} - \sum_{cyc}\frac{a}{b+c}\right) \overset{Nesbitt}{\le} \frac{r}{2R}\left(\frac{4R-2r}{r} - \frac{3}{2}\right)$$

$$= \frac{8R-7r}{4R} \overset{?}{\le} \frac{9R}{16r} \Leftrightarrow 9R^2-32Rr+28r^2 \overset{?}{\ge} 0 \Leftrightarrow (9R-14r)(R-2r) \overset{?}{\ge} 0 \to \text{true}$$

$$\because R \overset{Euler}{\ge} 2 \Rightarrow \sum_{cyc}\left(\frac{r_a}{b+c}.\sin A\right) \le \frac{9R}{16r} \text{ and so, } \frac{9}{8} \ge \sum_{cyc}\left(\frac{r_a}{b+c}.\sin A\right) \le \frac{9R}{16r}$$

$\forall$ Δ ABC, with equality iff Δ ABC is equilateral (QED)